INVERSION AS ROTATION IN ORDERED PITCH-CLASS SETS

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The phenomenon under investigation in this paper is nicely summarized by Joseph Straus in his discussion of the inversion operation in his book *Introduction to Post-Tonal Theory*: "*Generally* when you invert a set in normal form, the result will be the normal form of the new set written backwards. *There are many exceptions to this rule*, however, so beware!"[1] This paper investigates exactly how many exceptions there are to this "rule" and makes several observations as to why some sets behave this way while others do not.

Before we begin, a brief note on some notational conventions is in order. As this paper deals with the *ordering* of pitch-class sets and intervals, I will be using square brackets [ ] throughout the paper to indicate an ordered set of pitches or pitch classes and angled brackets < > to indicate an ordered set of intervals or interval classes.

The rotational order of a pitch-class set is not affected by transposition. A major triad in root position, for example, will continue to be in root position when transposed up or down by any interval. The same would thus be true of a triad in first or second inversion as well. Therefore, transposition can be said to be an order-preserving operation. In pitch-class set terms, while transposition will change the pitch-class content of a set, it will not change the set's intervallic content or the order in which those intervals occur because adding the same number to a pair of numbers will never change the distance between them as long as the added number is not negative: $(y - z) = (x + y) - (x + z)$ where $x \geq 0$. Thus, the transposition of a set in normal form will always remain in normal form.

Interestingly, although inversion does re-order a set and, in many cases, even produces an altogether different set (in cases where the set is not symmetrical), the inversion of a set in normal form is quite often the retrograde of the new set's normal form. That is, the inverted set is simply in normal form written backwards. Inverting a root-position C-major triad (which

[1] Joseph N. Straus, *Introduction to Post-Tonal Theory*, 3rd ed. (Upper Saddle River: Pearson, 2000), 47, emphasis mine.

happens to be normal form for this set class) at I_0, for example, returns a root-position F-minor triad read backwards: $I_0([0,4,7]) = [0,8,5]$. Thus, even though the inversion process produces a different set, rotational order is preserved. This can be seen even more clearly with symmetrical sets that produce the same set when inverted. Inverting a C-diminished triad at I_0, for example, gives us a root-position F#-diminished triad in reverse order: $I_0([0,3,6]) = [0,9,6]$. For both of these cases, because transposition is an order-preserving operation and since I_n is the same as I_0 followed by T_n, inversion at *any* index will always produce a triad in retrograded normal form. To generalize this behavior, let us define a function, N, that puts a pitch-class set in normal form and another function, R, that retrogrades an ordered pitch class set. In these terms, the behavior we have seen with the major/minor and diminished triads can be written as: $R(I_n(X)) = N(I_n(X))$. Sets for which this is true will be said to have normal-form symmetry (NFS).

As Straus noted earlier, however, all sets do not exhibit NFS. For example, inverting [0,2,7] at I_2 produces the very same pitch classes, but *not* in retrograde order: $I_2([0,2,7]) = [2,0,7]$. Inverting this set thus produces a *rotation* of the normal form rather than its retrograde. In such cases there is thus a conflation of inversion with rotation. Defining P_n as the function that permutes an ordered set by moving each member *n* order places to the left allows us to generalize this situation as: $P_n(R(I_n(X))) = N(I_n(X))$. Table 1 examines the relationship between inversion and rotation within the twelve trichord set classes.

Forte Number	Rotations	$R(I_0(\text{Rotations}))$	Mappings
3-1	Norm. = [0,1,2] P_1 = [1,2,0] P_2 = [2,0,1]	Norm. = [10,11,0] P_2 = [0,10,11] P_1 = [11,0,10]	Norm. ⇔ Norm. P_1 ⇔ P_2 P_2 ⇔ P_1
3-2	Norm. = [0,1,3] P_1 = [1,3,0] P_2 = [3,0,1]	Norm. = [9,11,0] P_2 = [0,9,11] P_1 = [11,0,9]	Norm. ⇔ Norm. P_1 ⇔ P_2 P_2 ⇔ P_1
3-3	Norm. = [0,1,4] P_1 = [1,4,0] P_2 = [4,0,1]	Norm. = [8,11,0] P_2 = [0,8,11] P_1 = [11,0,8]	Norm. ⇔ Norm. P_1 ⇔ P_2 P_2 ⇔ P_1
3-4	Norm. = [0,1,5] P_1 = [1,5,0] P_2 = [5,0,1]	Norm. = [7,11,0] P_2 = [0,7,11] P_1 = [11,0,7]	Norm. ⇔ Norm. P_1 ⇔ P_2 P_2 ⇔ P_1
3-5	Norm. = [0,1,6] P_1 = [1,6,0] P_2 = [6,0,1]	Norm. = [6,11,0] P_2 = [0,6,11] P_1 = [11,0,6]	Norm. ⇔ Norm. P_1 ⇔ P_2 P_2 ⇔ P_1
3-6	Norm. = [0,2,4] P_1 = [2,4,0] P_2 = [4,0,2]	Norm. = [8,10,0] P_2 = [0,8,10] P_1 = [10,0,8]	Norm. ⇔ Norm. P_1 ⇔ P_2 P_2 ⇔ P_1
3-7	Norm. = [0,2,5] P_1 = [2,5,0] P_2 = [5,0,2]	Norm. = [7,10,0] P_2 = [0,7,10] P_1 = [10,0,7]	Norm. ⇔ Norm. P_1 ⇔ P_2 P_2 ⇔ P_1
3-8	Norm. = [0,2,6] P_1 = [2,6,0] P_2 = [6,0,2]	Norm. = [6,10,0] P_2 = [0,6,10] P_1 = [10,0,6]	Norm. ⇔ Norm. P_1 ⇔ P_2 P_2 ⇔ P_1
3-9	Norm. = [0,2,7] P_1 = [2,7,0] P_2 = [7,0,2]	P_2 = [5,10,0] P_1 = [0,5,10] Norm. = [10,0,5]	Norm. ⇔ P_2 P_1 ⇔ P_1 P_2 ⇔ Norm.
3-10	Norm. = [0,3,6] P_1 = [3,6,0] P_2 = [6,0,3]	Norm. = [6,9,0] P_2 = [0,6,9] P_1 = [9,0,6]	Norm. ⇔ Norm. P_1 ⇔ P_2 P_2 ⇔ P_1
3-11	Norm. = [0,3,7] P_1 = [3,7,0] P_2 = [7,0,3]	Norm. = [5,9,0] P_2 = [0,5,9] P_1 = [9,0,5]	Norm. ⇔ Norm. P_1 ⇔ P_2 P_2 ⇔ P_1
3-12	Norm. = [0,4,8] P_1 = [4,8,0] P_2 = [8,0,4]	P_1 = [4,8,0] Norm. = [0,4,8] P_2 = [8,0,4]	Norm. ⇔ P_1 P_1 ⇔ Norm. P_2 ⇔ P_2

Table 1. The three rotations of each set class (normal, P_1(normal), and P_2(normal)) and the rotation of the inverted set they map to one another under RI_0.

As can be seen in the far-right column, each of these sets is "rotationally symmetrical" about one of its three rotations, while the remaining two rotations exchange with one another. It is important to note, however, that the two sets that do *not* exhibit NFS (3-9 and 3-12) are still rotationally symmetric, but this symmetry is not about the normal form.[2] Set class 3-12 (the augmented triad) is a rather unusual case, however, because this set is completely symmetrical—meaning that the intervallic content of each rotation is identical. In cases like these, Rahn's algorithm arbitrarily designates normal form as the "ordering with the *smallest initial pc number*," which in this case means that [0,4,8] is arbitrarily chosen over [8,0,4] because 0 is smaller than 8.[3] If one were to begin with [1,5,9], however, the retrograde of the inversion ([3,7,11]) *would*, in fact, be the normal form since 3 is the smallest initial pc number. Because of this lack of consistency, it seems that the concept of normal form is not particularly relevant for perfectly-symmetrical sets like 3-12. These sets will therefore be considered trivial cases of NFS and largely ignored for the remainder of the paper. Set class 3-9, then, is the only non-trivial case of a non-NFS trichord.

The ordering of a set under inversion is most directly applicable to the process of finding a pitch-class set's prime form. Here, one must first find the normal form of the set under consideration and then compare it to the normal form of its inversion to find the most compact and thus "best" normal form. For sets that possess NFS, one can simply retrograde the inverted set in order to find normal form, which saves a rather significant amount of time when one is dealing with sets of larger cardinalities. This begs the question, then, of whether there might be some way to quickly determine whether or not a set possess NFS without having to invert it and

[2] While it is beyond the scope of this paper to construct tables analogous to Table 1 for all the remaining set classes, it should be noted that sets of odd cardinality will always be symmetrical about at least one rotation simply because there is no way to group an odd number of elements into pairs without pairing an element with itself. Even-cardinality sets, on the other hand, can be non-rotationally symmetric, see, for example, 4-6, 4-19, and 4-24.

[3] John Rahn, *Basic Atonal Theory* (New York: Schirmer, 1980), 37.

test its retrograde against the correct normal form. The information provided in Table 1 does not immediately suggest why set class 3-9 does not exhibit NFS while the set classes that are very similar to it (3-8, 3-10, and 3-11) do exhibit NFS. So, to begin investigating this question, Table 2 displays all 50 non-trivial non-NFS sets along with their respective prime forms, interval vectors, degrees of symmetry,[4] $CINT_1$,[5] and the relation of the retrograded inversion to the inversion in normal form.

These 50 non-NFS sets make up 24% of the 208 total set classes (not counting sets of cardinality 0, 1, 2, 10, 11, or 12). Of these 50 sets, there is a single trichord, 4 tetrachords, 7 pentachords, 12 hexachords, 10 heptachords, 10 octachords, and 6 nonachords. Included among these are several historically significant sets like the major-major seventh chord (4-20), the anhemitonic pentatonic scale (5-35), the Schoenberg hexachord (6-z44), and the diatonic scale (7-35), to name only a few. The sets generally tend have index numbers toward the latter half of the total number of sets for that cardinality, meaning they tend to be more “spread out.” The earlies index number is 8-3 and the latest 6-z50, the last set of cardinality 6. The complement of a non-NFS set is not necessarily a non-NFS set nor is the z-partner of a non-NFS set necessarily a non-NFS set. 33 (66%) of these sets are inversionally symmetric and only one set (8-25) is transpositionally symmetric at an index other than 0. 8-25 is also the only set that is inversionally symmetric about more than one axis. The maximum distance the retrograded inversion ever lies from the normal form is 4 permutations. Nearly half (24) of the sets lie a single permutation from the normal form, 16 lie 2 permutations away, 5 lie 3 permutations away, and 5 lie 4 permutations away.

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[4] The degrees of symmetry of a pitch-class set is an ordered pair in which the first value indicates the number of transpositional values under which the set will map onto itself and the second value indicates the number of inversional values under which the set will map onto itself.

[5] The interval between each adjacent pitch class within a set including the “wraparound interval” from the last pitch class to the first. See Robert Morris, *Composition with Pitch-Classes: A Theory of Compositional Design* (New Haven: Yale University Press, 1987), 40–1.

Forte Number	Prime Form	Interval Vector	Symmetry	CINT$_1$	Relation of R to N
3-9	[0,2,7]	<010020>	(1,1)	<2,5,5>	$P_1(R) = N$
4-6	[0,1,2,7]	<210021>	(1,1)	<1,1,5,5>	$P_1(R) = N$
4-19	[0,1,4,8]	<101310>	(1,0)	<1,3,4,4>	$P_1(R) = N$
4-20	[0,1,5,8]	<101220>	(1,1)	<1,4,3,4>	$P_2(R) = N$
4-24	[0,2,4,8]	<020301>	(1,1)	<2,2,4,4>	$P_1(R) = N$
5-13	[0,1,2,4,8]	<221311>	(1,0)	<1,1,2,4,4>	$P_1(R) = N$
5-15	[0,1,2,6,8]	<220222>	(1,1)	<1,1,4,2,4>	$P_2(R) = N$
5-z17	[0,1,3,4,8]	<212320>	(1,1)	<1,2,1,4,4>	$P_1(R) = N$
5-31	[0,1,3,6,9]	<114112>	(1,0)	<1,2,3,3,3>	$P_2(R) = N$
5-32	[0,1,4,6,9]	<113221>	(1,0)	<1,3,2,3,3>	$P_1(R) = N$
5-34	[0,2,4,6,9]	<032221>	(1,1)	<2,2,2,3,3>	$P_1(R) = N$
5-35	[0,2,4,7,9]	<032140>	(1,1)	<2,2,3,2,3>	$P_2(R) = N$
6-27	[0,1,3,4,6,9]	<225222>	(1,0)	<1,2,1,2,3,3>	$P_1(R) = N$
6-z28	[0,1,3,5,6,9]	<224322>	(1,1)	<1,2,2,1,3,3>	$P_1(R) = N$
6-z37	[0,1,2,3,4,8]	<432321>	(1,1)	<1,1,1,1,4,4>	$P_1(R) = N$
6-z38	[0,1,2,3,7,8]	<421242>	(1,1)	<1,1,1,4,1,4>	$P_2(R) = N$
6-z42	[0,1,2,3,6,9]	<324222>	(1,1)	<1,1,1,3,3,3>	$P_2(R) = N$
6-z44	[0,1,2,5,6,9]	<313431>	(1,0)	<1,1,3,1,3,3>	$P_1(R) = N$
6-z45	[0,2,3,4,6,9]	<234222>	(1,1)	<2,1,1,2,3,3>	$P_1(R) = N$
6-z46	[0,1,2,4,6,9]	<233331>	(1,0)	<1,1,2,2,3,3>	$P_1(R) = N$
6-z47	[0,1,2,4,7,9]	<233241>	(1,0)	<1,1,2,3,2,3>	$P_2(R) = N$
6-z48	[0,1,2,5,7,9]	<232341>	(1,1)	<1,1,3,2,2,3>	$P_3(R) = N$
6-z49	[0,1,3,4,7,9]	<224322>	(1,1)	<1,2,1,3,2,3>	$P_2(R) = N$
6-z50	[0,1,4,6,7,9]	<224232>	(1,1)	<1,3,2,1,2,3>	$P_4(R) = N$
7-10	[0,1,2,3,4,6,9]	<445332>	(1,0)	<1,1,1,1,2,3,3>	$P_1(R) = N$
7-z12	[0,1,2,3,4,7,9]	<444342>	(1,1)	<1,1,1,1,3,2,3>	$P_2(R) = N$
7-16	[0,1,2,3,5,6,9]	<435432>	(1,0)	<1,1,1,2,1,3,3>	$P_1(R) = N$
7-z17	[0,1,2,4,5,6,9]	<434541>	(1,1)	<1,1,2,1,1,3,3>	$P_1(R) = N$
7-19	[0,1,2,3,6,7,9]	<434343>	(1,0)	<1,1,1,3,1,2,3>	$P_3(R) = N$
7-21	[0,1,2,4,5,8,9]	<424641>	(1,0)	<1,1,2,1,3,1,3>	$P_2(R) = N$
7-22	[0,1,2,5,6,8,9]	<424542>	(1,1)	<1,1,3,1,2,1,3>	$P_4(R) = N$
7-33	[0,1,2,4,6,8,10]	<262623>	(1,1)	<1,1,2,2,2,2,2>	$P_4(R) = N$
7-34	[0,1,3,4,6,8,10]	<254442>	(1,1)	<1,2,1,2,2,2,2>	$P_3(R) = N$
7-35	[0,1,3,5,6,8,10]	<254361>	(1,1)	<1,2,2,1,2,2,2>	$P_2(R) = N$
8-3	[0,1,2,3,4,5,6,9]	<656542>	(1,1)	<1,1,1,1,1,1,3,3>	$P_1(R) = N$
8-7	[0,1,2,3,4,5,8,9]	<645652>	(1,1)	<1,1,1,1,1,3,1,3>	$P_2(R) = N$
8-8	[0,1,2,3,4,7,8,9]	<644563>	(1,1)	<1,1,1,1,3,1,1,3>	$P_3(R) = N$
8-21	[0,1,2,3,4,6,8,10]	<474643>	(1,1)	<1,1,1,1,2,2,2,2>	$P_3(R) = N$
8-22	[0,1,2,3,5,6,8,10]	<465562>	(1,0)	<1,1,1,2,1,2,2,2>	$P_2(R) = N$
8-23	[0,1,2,3,5,7,8,10]	<465472>	(1,1)	<1,1,1,2,2,1,2,2>	$P_4(R) = N$
8-24	[0,1,2,4,5,6,8,10]	<464743>	(1,1)	<1,1,2,1,1,2,2,2>	$P_2(R) = N$
8-25	[0,1,2,4,6,7,8,10]	<464644>	(2,2)	<1,1,2,2,1,1,2,2>	$P_1(R) = N$
8-26	[0,1,3,4,5,7,8,10]	<456562>	(1,1)	<1,2,1,1,2,1,2,2>	$P_1(R) = N$
8-27	[0,1,2,4,5,7,8,10]	<456553>	(1,0)	<1,1,2,1,2,1,2,2>	$P_1(R) = N$

9-6	[0,1,2,3,4,5,6,8,10]	<686763>	(1,1)	<1,1,1,1,1,1,2,2,2>	$P_2(R) = N$
9-7	[0,1,2,3,4,5,7,8,10]	<677673>	(1,0)	<1,1,1,1,1,2,1,2,2>	$P_1(R) = N$
9-8	[0,1,2,3,4,6,7,8,10]	<676764>	(1,0)	<1,1,1,1,2,1,1,2,2>	$P_1(R) = N$
9-9	[0,1,2,3,5,6,7,8,10]	<676683>	(1,1)	<1,1,1,2,1,1,1,2,2>	$P_1(R) = N$
9-10	[0,1,2,3,4,6,7,9,10]	<668664>	(1,1)	<1,1,1,1,2,1,2,1,2>	$P_4(R) = N$
9-11	[0,1,2,3,5,6,7,9,10]	<667773>	(1,0)	<1,1,1,2,1,1,2,1,2>	$P_2(R) = N$

Table 2. The 50 non-trivial non-NFS sets along with their prime forms, interval vectors, degrees of symmetry, CINTS_1, and the relationship of the retrograded inversion to the inversion in normal form.

The only characteristic shared by all 50 sets is that the "wraparound" interval (the last position in the CINT_1)—which is, by default, also the largest interval since these sets are in normal form—is also found as an internal interval. In most cases, the number of permutations needed to reach the normal form of the inverted set is directly related to the distance between the wraparound interval and the internal interval of the same size. 4-20, for example, is the first set requiring two permutations to reach normal form and it is also the first set in which the wraparound interval and the internal interval of the same size are non-adjacent and thus are two "steps" apart. Similarly, 6-z48 is the first set requiring 3 permutations to reach normal form and the wraparound interval and the internal interval of the same size are 3 "steps" apart. Sets with more than one internal interval that is the same size as the wraparound interval are exceptions to this general principle however (see 5-31 for the first instance).

Both Scotto and Kostka also make note of sets in which the wraparound interval is also an internal interval and the effect of this phenomenon on normal order.[6] This alone cannot be the deciding characteristic of NFS, however, because there are several non-trivial sets exhibiting this same feature that *are* NFS. These sets are listed in Table 3.

[6] See Ciro Scotto, "Normal Form, Successive Interval Arrays, Transformations, and Set Classes: A Re-Evaluation and Reintegration," in *Mathematics and Computation in Music: First International Conference*, ed. Timour Klouche and Thomas Noll (Berlin: Springer, 2009), 25–51; Stefan M. Kostka, *Materials and Techniques of Twentieth-Century Music* (Upper Saddle River: Prentice Hall, 1999), 182.

Forte Number	Prime Form	Interval Vector	Symmetry	CINT$_1$
4-9	[0,1,6,7]	<200022>	(2,2)	<1,5,1,5>
4-25	[0,2,6,8]	<020202>	(2,2)	<2,4,2,4>
5-20	[0,1,5,6,8]	<211231>	(1,0)	<1,4,1,2,4>
6-7	[0,1,2,6,7,8]	<420243>	(2,2)	<1,1,4,1,1,4>
6-20	[0,1,4,5,8,9]	<303630>	(3,3)	<1,3,1,3,1,3>
6-z29	[0,2,3,6,7,9]	<224232>	(1,1)	<2,1,3,1,2,3>
6-30	[0,1,3,6,7,9]	<224223>	(2,0)	<1,2,3,1,2,3>
6-31	[0,1,4,5,7,9]	<223431>	(1,0)	<1,3,1,2,2,3>
7-z18	[0,1,4,5,6,7,9]	<434442>	(1,0)	<1,3,1,1,1,2,3>
7-20	[0,1,2,5,6,7,9]	<433452>	(1,0)	<1,1,3,1,1,2,3>
8-9	[0,1,2,3,6,7,8,9]	<644464>	(2,2)	<1,1,1,3,1,1,1,3>
8-28	[0,1,3,4,6,7,9,10]	<448444>	(4,4)	<1,2,1,2,1,2,1,2>
9-12	[0,1,2,4,5,6,8,9,10]	<666963>	(3,3)	<1,1,2,1,1,2,1,1,2>

Table 3. Non-trivial NFS sets in which the "wraparound" interval is also an internal interval.

While it is true that several of these sets exhibit much higher degrees of symmetry than the sets in Table 2, it is still not clear exactly what (if anything) differentiates these sets from those in Table 3. Interestingly, four of these sets are members of the unfortunate group of sets that receive different prime forms under Forte's and Rahn's normal-form algorithms: 5-20, 6-z29, 6-31, and 7-20. Even under Forte's algorithm, though, these sets still exhibit NFS. This suggests that this property has something to do with the inherent limitations of normal-form algorithms or perhaps even the concept of normal form in general.[7]

[7] See Scotto, "Normal Form."

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