EXPLORING COHN'S SUM-CLASS SYSTEM THROUGH CHARLES VILLIERS STANFORD'S *LA BELLE DAME SANS MERCI*

David Orvek

Richard Cohn's voice-leading model known as the "sum-class" system is an enormously valuable analytical tool because it allows for the generalization of the voice leading between any major or minor triad.[1] Such a model is useful in the analysis of the large repertoire of triadic music in which "root-defined continuity [is pushed] farther and farther into the background, leaving voice leading as the sole means of foreground progression."[2] But Cohn's model accomplishes this generalization in a way that is counterintuitive to our usual conceptualization of interval and transposition within a pitch space and, furthermore, obscures the fundamental equivalence of harmonic progressions in major and minor keys. The shortcomings of this way of thinking are made quite evident by a pair of harmonic progressions found in Charles Villiers Stanford's *La bell Dame sans merci* that are exactly equivalent from a voice-leading perspective but are interpreted differently within Cohn's model. In this paper, I expand Cohn's sum-class system through the group-theoretical notion of a commuting group in order to make the system's analytical findings are more musically intuitive and applicable to a wider variety of contexts.

The Sum-Class System

Since the work of Richard Cohn and Robert Cook so nicely explores the formal and philosophical development of the sum-class system, only the briefest of overviews need concern us here. The reader is encouraged, therefore, to read the accounts of Cohn and Cook for a more comprehensive perspective.[3]

[1] Richard Cohn, "Square Dances with Cubes," *Journal of Music Theory* 42, no. 2 (1998): 283-96.

[2] Jonathan D. Kramer, "New Temporalities in Music," *Critical Inquiry* 7, no. 3 (1981): 541.

[3] See Cohn, "Square Dances with Cubes" and Robert Cook, "Transformational Approaches to Romantic Harmony and the Late Works of César Franck" (Ph.D. diss., University of Chicago): 60–107.

We begin by defining a function, PCSUM, that takes as its argument any triad and returns the summed value of that triad's three constituent pitch classes (where C = 0). Cohn defines this formally as:

Definition 1. PCSUM(X) = $\sum_{n=1}^{3} x_n$ (mod 12).[4]

Applying this function to each of the 24 consonant triads (the major and minor triads, notated as X+ and X- from now on) yields only 8 different PCSUM values: 1, 2, 4, 5, 7, 8, 10, and 11, which is to say that several triads have the same PCSUM value. In fact, this overlap is very consistent: triads of the same quality whose roots lie a major third apart (like G+, B+, and Eb+) have the same PCSUM value. From this, we may define an equivalence relation that partitions the 24 triads into 8 equivalence classes known as "sum classes":

Definition 2. Let *S* be the set of all 24 consonant triads and *R* a subset of *S* such that *a*, *b* ∈ *R* for all *a, b* ∈ *S* if PCSUM(*a*) = PCSUM(*b*).

Theorem 1. *R* is an equivalence relation.

Proof: Because membership in *R* is defined using the traditional notion of equality, it should be quite clear that PCSUM(*a*) = PCSUM(*a*) for all *a* ∈ *S*; similarly that if PCSUM(*a*) = PCSUM(*b*) then PCSUM(*b*) = PCSUM(*a*) for all *a, b* ∈ *S*; and finally, if PCSUM(*a*) = PCSUM(*b*) and PCSUM(*b*) = PCSUM(*c*), then PCSUM(*a*) = PCSUM(*c*) for all *a, b, c* ∈ *S*.

R is, therefore, an equivalence relation on *S* partitioning it into 8 equivalence classes. These classes will be hereafter known as sum classes and notated as:

Definition 3. $\boxed{\text{X}}$ is the class of triads *y* where PCSUM(*y*) = X.[5]

[4] Adapted from Definition 5 in Cohn, "Square Dances with Cubes," 286; all equations will be modulo 12 for the remainder of this paper unless otherwise specified.

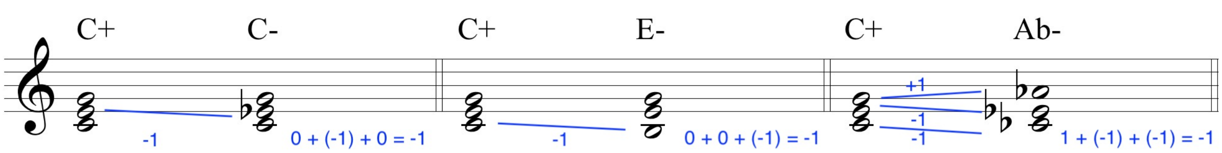
[5] Adapted from Definition 6 in Cohn, "Square Dances with

These classes and their members are shown in Table 1. Note that sum classes congruent to 1 modulo 3 (1, 4, 7, 10) contain minor triads while sum classes congruent to 2 modulo 3 (2, 5, 8, 11) contain major triads.

Sum Class	**Triadic Members**
1	{A-, F-, C#-}
2	{A+, F+, C#+}
4	{D-, F#-, Bb-}
5	{D+, F#+, Bb+}
7	{Eb-, G-, B-}
8	{Eb+, G+, B+}
10	{C-, E-, G#-}
11	{C+, E+, G#+}

Table 1. The eight "sum classes" and their triadic members

While potentially interesting, these sum classes are not particularly useful in and of themselves until we consider their relationship to voice leading. Observe, for example, the phenomenon seen in Example 1.



Example 1. The total number of half steps traversed from C+ to C-, E- and Ab-

This example measures the total voice-leading distance from C+ (a member of $\boxed{11}$) to C-, E-, and Ab- (members of $\boxed{10}$) as the sum of the *directed* intervals traversed by each voice. A directed interval takes the direction of the motion into account, with the result that same-interval motion in opposite directions sums to zero.[6] As can be seen in the example, the resulting value (which Cohn calls the Directed Voice-Leading Sum, see Definition 4) is the same in all three cases.

Definition 4. DVLS(X,Y) = $\sum_{n=1}^{3}(y_n - x_n)$.[7]

Recall that PCSUM calculates the summed value of the three members of a single triad and note that DVLS calculates the summed value of the total distance traveled by the three voices in the motion between two triads. Because both of these functions deal with the summed values of all three triadic members, their values may be interrelated. In fact, Cohn shows generally that DVLS(X,Y) = PCSUM(Y) – PCSUM(X),[8] which, in turn, means that DVLS($\boxed{X}$,$\boxed{Y}$) = Y – X.[9]

Cohn then defines a group of eight "sum-class transformations" to act on the sum class space {Y0, Y3, Y6, Y9, X1, X4, X7, X10} and defines their actions on the sum classes as:

Definition 5. OPn(s) = s + n if s ≡ 1 modulo 3; s – n if s ≡ 2 modulo 3.[10]

This defines the operations contextually so that they act differently upon sum classes containing minor (classes congruent to 1 mod 3) and major (classes congruent to 2 mod 3) triads. This results in the mappings seen in Table 2. Because of the relationship between sum class and directed voice-leading distance, we may also understand the actions of these transformations in

[6] Cohn, "Square Dances with Cubes," 285.

[7] Originally Definition 4 in Cohn, "Square Dances with Cubes," 285.

[8] Originally Theorem 1a in Cohn, "Square Dances with Cubes," 286.

[9] Originally Theorem 1b in Cohn, "Square Dances with Cubes," 287.

[10] Originally Definition 7 in Cohn, "Square Dances with Cubes," 288.

terms of the directed voice-leading distance between the triads contained in each sum class. Specifically, the Y*n* transformations relate sum classes whose constituent triads are the of the *same* mode that lie *n* semitones away if the two sum classes contain major triads or n^{-1} semitones if the sum classes contain minor triads. Similarly, the X*n* transformations relate sum classes whose constituent triads are of *different* mode that lie *n* or n^{-1} semitones from each other in directed voice-leading space. X1, for example, relates all the sum classes for which the directed voice-leading distance between any of their constituent triads is a single ascending or descending semitone. Y0, the identity of the group, is the trivial transformation that takes any sum class to itself.

Sum-Class Transformation	**Action on Sum Classes**
Y0	(1) (2) (4) (5) (7) (8) (10) (11)
Y3	(1, 4, 7, 10) (2, 11, 8, 5)
Y6	(1, 7) (2, 8) (4, 10) (5, 11)
Y9	(1, 10, 7, 4) (2, 5, 8, 11)
X1	(1, 2) (4, 5) (7, 8) (10, 11)
X4	(1, 5) (2, 10) (4, 8) (7, 11)
X7	(1, 8) (2, 7) (4, 11) (5, 10)
X10	(1, 11) (2, 4) (5, 7) (8, 10)

Table 2. Actions of the sum-class transformations on the sum classes

Because of the voice-leading consistency, the sum-class transformations might thus also be thought of as "intervals" in terms of a Generalized Interval System (GIS).[11] From this intervallic perspective, we might say that $\boxed{1}$ and $\boxed{7}$ (and also $\boxed{2}$ and $\boxed{8}$, $\boxed{4}$ and $\boxed{10}$, and $\boxed{5}$ and $\boxed{11}$) all lie the "interval" of Y6 from each other. This notion of interval actually becomes quite intuitive when we recall that the subscript of each sum-class transformation is equal to the total number of semitones traversed between chords contained within sum classes related by this transformation. Each of the three triads in $\boxed{1}$ is thus 6 total semitones away from each of the three triads in $\boxed{7}$ and so on.

The way these "intervals" compose with one another, however, is rather unintuitive. $\boxed{1}$ and $\boxed{2}$, for example, lie the "interval" of X1 from each other. Yet if we transform each of these sum classes by Y3, we get $\boxed{4}$ and $\boxed{11}$, which lie the "interval" of X7 from one another. This is because this group of sum-class transformations is non-commutative (or non-abelian), which also means that the order in which the operations are composed matters ($X1 \circ Y3 \neq Y3 \circ X1$). In such a group, the majority of transformations will behave like we saw above. In any group, however, there will be at least one transformation (the identity) that will *not* affect the intervals between every pair of sum classes. Lewin calls such transformations "interval-preserving" transformations.[12] There are two interval-preserving transformations for the sum-class group: Y0 and Y6. This can be seen in the Cayley table for the composition of these transformations, presented in Table 3.

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[11] See David Lewin, *Generalized Musical Intervals and Transformations* (Oxford: Oxford University Press, 2011).

[12] Lewin, *Generalized Musical Intervals and Transformations*, 48.

	Y0	Y3	Y6	Y9	X1	X4	X7	X10
Y0	Y0	Y3	Y6	Y9	X1	X4	X7	X10
Y3	Y3	Y6	Y9	Y0	X4	X7	X10	X1
Y6	Y6	Y9	Y0	Y3	X7	X10	X1	X4
Y9	Y9	Y0	Y3	Y6	X10	X1	X4	X7
X1	X1	X10	X7	X4	Y0	Y9	Y6	Y3
X4	X4	X1	X10	X7	Y3	Y0	Y9	Y6
X7	X7	X4	X1	X10	Y6	Y3	Y0	Y9
X10	X10	X7	X4	X1	Y9	Y6	Y3	Y0

Table 3. Cayley table of the composition of the sum-class transformations[13]

In a non-commutative GIS such as this one, Lewin tells us that there exists another group of transformations acting on the same space that will commute with every transformation in our first group.[14] We may find this commuting group through the algebraic method described in Satyendra.[15] Doing so results in the group of eight transformations seen in Table 4.[16] As can be seen, Z0 and Z6 correspond exactly with Y0 and Y6. This is to be expected since these transformations were already the interval-preserving transformations of the Y*n*/X*n* group.

[13] Also in Cook, "Transformational Approaches to Romantic Harmony," 101.

[14] See Lewin, *Generalized Musical Intervals and Transformations*, 48–50 and Ramon Satyendra, "An Informal Introduction to Some Formal Concepts from Lewin's Transformational Theory," *Journal of Music Theory* 48, no. 1 (2004): 131–4.

[15] Satyendra, "An Informal Introduction," 131–4.

[16] This same group (though labeled differently) is also discussed in Cook, "Transformational Approaches to Romantic Harmony," 103–5.

Commuting Transformations	**Action on Sum Classes**
Z0	(1) (2) (4) (5) (7) (8) (10) (11)
Z3	(1, 4, 7,10) (2, 5, 8, 11)
Z6	(1, 7) (2, 8) (4,10) (5, 11)
Z9	(1, 10, 7, 4) (2, 11, 8, 5)
W0	(1, 11) (2, 10) (4, 8) (5, 7)
W3	(1, 2) (4, 11) (5, 10) (7, 8)
W6	(1, 5) (2, 4) (7, 11) (8, 10)
W9	(1, 8) (2, 7) (4, 5) (10, 11)

Table 4. The commuting group of transformations also acting on the sum classes

It is very important for us to be mindful of the fact that both of these groups of sum-class transformations are defined to act on the sum classes themselves—as their name implies. It is thus only secondarily, by transforming the sum classes, that the sum-class transformations interact with the triads that are contained within each sum class. To say that C+ is mapped to G+ via Y9 is thus not technically true. Instead, it is $\boxed{11}$ as a whole that is mapped to $\boxed{8}$ as a whole, while C+ and G+ are only representatives of $\boxed{11}$ and $\boxed{8}$.[17]

To speak correctly of the mappings from triad to triad, we must, therefore, invoke transformations defined to act upon triads. The so called “neo-Riemannian” transformations are one such groups of transformations. Cohn, in fact, notes that the sum-class transformations act as equivalence classes on neo-Riemannian transformations.[18] This relationship may be defined formally as:

[17] I am indebted to David Clampitt for directing my attention to this important distinction.

[18] Cohn, “Square Dances with Cubes,” 289.

Definition 6. Let *S* be the set of all 24 consonant triads (set class 3-11), *T* the set of 24 neo-Riemannian transformations, and *R* a relation on *T* such that $s, t \in R$ for all $s, t \in T$ if PCSUM($s(a)$) = PCSUM($t(a)$) for all $a \in S$.

R can be proven to be an equivalence relation via the same process seen in theorem 1. That is to say that two neo-Riemannian transformations are equivalent to one another under *R* if their products belong to the same sum class. These equivalence relations can be seen in Table 5.[19]

Y*n*/X*n* Sum-Class Transformation	**Neo-Riemannian Transformations**
Y0	{E, PL, LP}
Y3	{RP, RL, RPLP}
Y6	{LRPR, RPRP, RPRL}
Y9	{PLPR, LR, PR}
X1	{P, L, H (PLP)}
X4	{PRP, LRP (PRL), LRL}
X7	{RLR, RPR, RPLPR}
X10	{R, S[20] (LPR or RPL), N[21] (PLR or RLP)}

Table 5. The eight Y*n*/X*n* sum-class transformations and their neo-Riemannian members

Since the Z*n*/W*n* group permutes the sum classes differently than the Y*n*/X*n* group, we might assume that the Z*n*/W*n* group would not be able to act as equivalence classes on the neo-Riemannian transformations. Indeed, the actions of the P (or parallel) transformation

[19] These operations are written using the conventions of left-functional orthography, meaning that they are read and performed right-to-left.

[20] Cohn, "Square Dances," 290: Known as the "slide" operation, which inverts a triad about its third.

[21] Cohn, "Square Dances," 290: Weitzmann's "Nebenverwandt" operation, which invers a triad about its Riemannian root.

demonstrates this nicely. P exchanges any triad with its parallel major or minor triad. Referring again to Table 1, we note that parallel major and minor triads are always located in adjacent sum classes (1 and 2, 4 and 5, 7 and 8, and 10 and 11). As can be seen in Table 4, however, there is no single Z*n* or W*n* transformation that exchanges these pairs of sum classes. There is thus no way to interrelate the actions of the Z*n*/W*n* group with the actions of the neo-Riemannian group.

Might there be some other group of triadic transformations that could be contained within the Z*n*/W*n* group like the neo-Riemannian group is within the Y*n*/X*n* group? Since we know that the Z*n*/W*n* group is the commuting group (or dual) of the Y*n*/X*n* group, we might begin by looking for the neo-Riemannian group's own commuting group. Satyendra shows that the T_n/I_n group (when defined to act on triads) is the commuting group of the neo-Riemannian group.[22] To see if this group does in fact interact with the Z*n*/W*n* group, we must calculate the action of each T_n and I_n transformation on the 24 triads. T_1, for example, maps any triad to the same-quality triad a semitone above it. In Table 1, we find that T_1 of any triad located in X is always located in X + 3. From Table 4 it can be seen that Z3 relates pairs of sum classes that stand in exactly this relationship. The actions of the other 23 members of the T_n/I_n group also correspond to the actions of a member of the Z*n*/W*n* group. The Z*n*/W*n* transformations can thus act as equivalence classes on the T_n/I_n group just as the Y*n*/X*n* group for the neo-Riemannian group, and an equivalence relation can be defined formally as in Definition 6 (this is left to the reader). Table 6 shows this relationship.

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[22] Satyendra, "An Informal Introduction," 118–23.

Z*n*/W*n* Sum-class Transformation	**T_n/I_n Transformations**
Z0	$\{T_0, T_4, T_8\}$
Z3	$\{T_1, T_5, T_9\}$
Z6	$\{T_2, T_6, T_{10}\}$
Z9	$\{T_3, T_7, T_{11}\}$
W0	$\{I_0, I_4, I_8\}$
W3	$\{I_1, I_5, I_9\}$
W6	$\{I_2, I_6, I_{10}\}$
W9	$\{I_3, I_7, I_{11}\}$

Table 6. The eight Z*n*/W*n* sum-class transformations and their T_n/I_n members

As with the Y*n* and X*n* transformations, the actions of the Z*n* transformations are directly related to the DVLS between the sum classes they relate. Table 4 reveals, however, that the DVLS values between sum classes related by the W*n* transformations are *not* consistent. For example, DVLS(1,11) = 10, while DVLS(2,10) = 8 though both pairs are related by W0. This is because the I_n transformations contained within each W*n* sum-class transformation are defined in reference to a pitch axis, rather than members of a triad. Whereas P exchanges all major and minor triads that share a root and a fifth, I_0 exchanges all triads who are mirrored around pitch class 0. The I_n transformations and the W*n* sum-class transformations that contain them are thus *unable* to generalize directed voice-leading distance.

The Y*n*, X*n*, and Z*n* sum-class transformations, however, *are* consistent with respect to directed voice-leading distance. What these structures allow us to do, then, is to understand groups of transformations as equivalent from a voice-leading perspective. That is to say, any of the three triadic transformations contained within a single Y*n*, X*n*, or Z*n* transformation all transform triads by the same number of half steps. Satyendra notes, however, that it is important for us to keep in mind that just because these two groups commute with one another does not mean that there is a one-to-one mapping between them; they are isomorphic to one another, but this does not mean that their actions are the same.[23]

Since both the Y*n* and X*n* transformations can be used to generalize voice leading between all 24 triads and both belong to the same group, why bother invoking the commuting group of Z*n* and W*n* transformations and their T_n/I_n members at all? Indeed, Cohn seems quite content to only work with the Y*n*/X*n* group in "Square Dances with Cubes." The reason lies in the contextual definition of the Y*n* and X*n* transformations as seen in Definition 5. Recall that this contextual definition caused the transformations to act differently upon sum classes containing major triads than those containing minor triads. This is all well and good for Cohn since he is mostly interested in maximally-smooth voice leading (that is, motion by a single half or whole step), which always involves alternating major and minor triads. Yet it is not fair to assume that such maximally-smooth progressions are the only way to systematically traverse voice-leading space or similarly that these are the only types of progressions composers are interested in. As we will see below, progressions of same-mode triads can also produce consistent voice leading. It is here that the Y*n*/X*n* group falls short because the triadic transformations contained within them are the contextual neo-Riemannian transformations.

Such contextually defined transformations act very intuitively when moving between triads of different mode because *we normally conceive of the "distances" between such triads in contextual terms*. The R transformation, for example, takes any triad to its relative major or minor. We perceive this "distance" or relationship to be the same for all pairs of triads even though the "underlying" I_n transformation changes from one context to the next. For example,

[23] Satyendra, "An Informal Introduction," 122.

C+ to A- is I_4 while the "same" R transformation from C#- to E+ is I_0. We thus expect the same neo-Riemannian transformation to move differently across the sum-class space depending on the quality of the triad it is applied to.

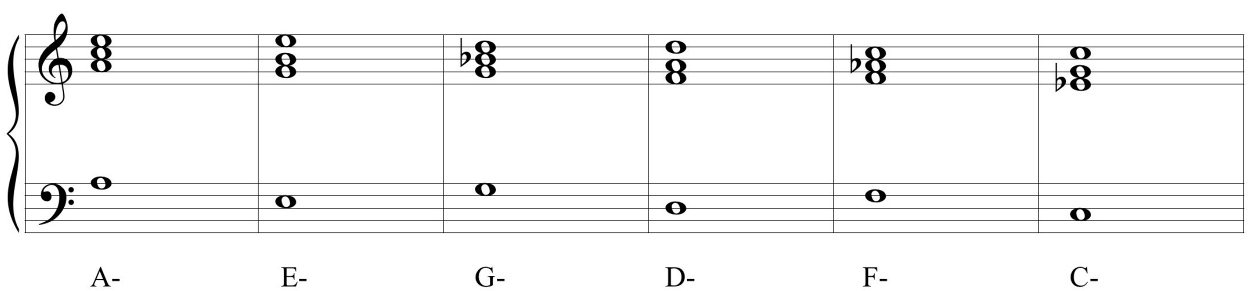
Neo-Riemannian transformations do not provide an intuitive notion of interval between triads of the same mode, however, because we usually think of this distance in terms of a transposition, *which only moves in a single direction*. It thus seems strange to us that the same operation should transpose a major triad up nine half steps while transposing a minor up only three (PR(C+) = A+ whereas PR(C-) = Eb-). Conversely, this means that the same "interval," as we would normally conceive it, receives *complementary* neo-Riemannian transformations for major and minor chords (A- to E- is LR while A+ to E+ is RL).

The Z*n* transformations, on the other hand, contain T_n transformations, whose actions *do* correspond with our usual notion of an interval between two same-mode triads. It thus seems to me that the strongest and most useful analytical model is achieved by utilizing the X*n* transformations for situations in which the harmonic motion is between triads of different mode and the Z*n* transformations for progressions of same-mode triads.[24] Furthermore, since these two groups commute with one another, their analytic findings may be interrelated with one another in interesting ways.

[24] I am not the first to suggest or use such a mixing of groups. Robert Cook makes frequent use of all four sum-class transformations in all sorts of combinations. See Cook, "Transformational Approaches to Romantic Harmony."

Analysis of Charles Villiers Stanford's *La belle Dame sans merci*

We now turn to the application of this system to a piece by Charles Villiers Stanford, which will allow us to see its analytic power and also to explore more of its possibilities. A harmonic reduction of measures 98–108 from Stanford's *La belle Dame sans merci* is presented in Example 2. Note first of all that the progression (referred to as "progression 1" from now on) consists entirely of same-mode triads. Because of this, we may consider the distance between each "node" in progression 1 in terms of a transpositional interval or a transformation that takes us from one node to the next. Doing so results in the transformational network seen in Figure 1. This network reveals that the entirety of progression 1 is generated by the alternation of two transpositions (T_7/T_3) and that each iteration of this transpositional pair results in a descent by whole step. A long-range T_3 also bookends progression 1.



Example 2. Harmonic reduction of mm. 98–108, "progression 1"[25]

[25] In the actual voicing of the C- chord in measure 108 the Eb is in the soprano and the G is omitted altogether. The voicing presented here is that implied by the continuation of the sequence.

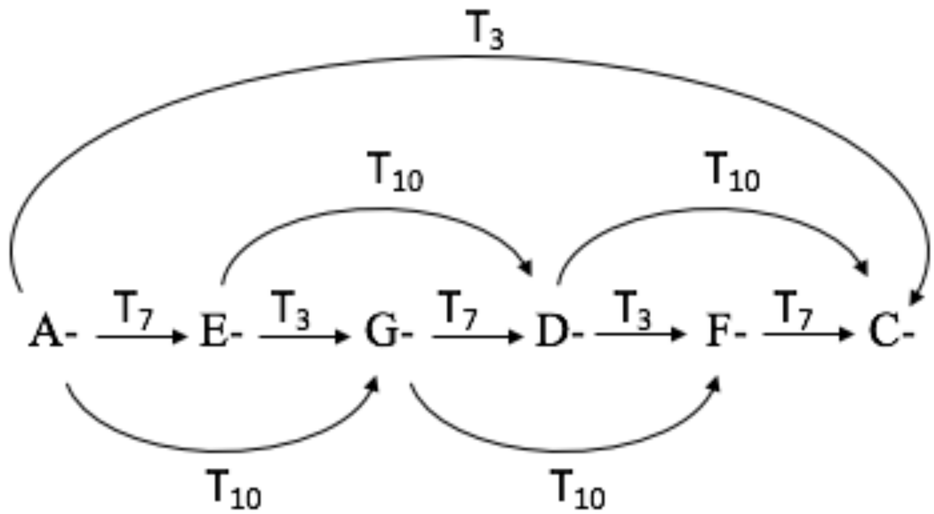


Figure 1. The transpositional transformations relating each chord in progression 1

Interestingly, though progression 1 is generated by two different transpositions, the voice leading from chord to chord remains consistent. Each triad has one tone in common with the triads on either side of it (those related by T_7 and T_3) and this common tone alternates between the soprano and tenor voices (see Example 2). When not holding a common tone, the soprano and tenor descend by whole step. The alto, on the other hand, descends by half step between each chord. The combination of this half step with the whole step of the soprano or tenor results in a total voice-leading distance of three half steps between each chord. Z9 contains all transformations that result in voice-leading descent by three half steps. This should not surprise us since T_3 and T_7 are contained within Z9 (see Table 6).

We may thus relabel Figure 1 in terms of sum classes and sum-class transformations (see Figure 2). This figure reveals that the entire surface of progression 1 is generated solely by Z9. Also revealed by the figure is the result of successively composing Z9 with itself. Each consecutive iteration of Z9 produces a new Z relation with the first triad: $(Z9)^1 = Z9$; $(Z9)^2 = Z6$; $(Z9)^3 = Z3$; $(Z9)^4 = Z0$; $(Z9)^5 = Z9$. A Z9 transformation thus relates the two ends of progression 1 as well as each pair of chords within it. In fact, progression 1 composes Z9 with itself exactly the number of times needed to produce Z9 again. We may generalize the composition of any Z transformation with itself as:

Definition 7. $(Z_x)^n = Z_{(x)(n)}$.

Given a succession of triads generated by a single transformation Z*n*, we may calculate a Sum-Class Transformation Interval (SCTI) between any two order positions M and N in that succession via:

Definition 8. SCTI(M,N) = $(Zn)^{(N-M)}$.

Returning to Figure 2, we may verify that SCTI(1,2) = $(Z9)^1$ = Z9; SCTI(1,3) = $(Z9)^2$ = Z6: SCTI(1,4) = $(Z9)^3$ = Z3; SCTI(1,5) = $(Z9)^4$ = Z0: SCTI(1,6) = $(Z9)^5$ = Z9.

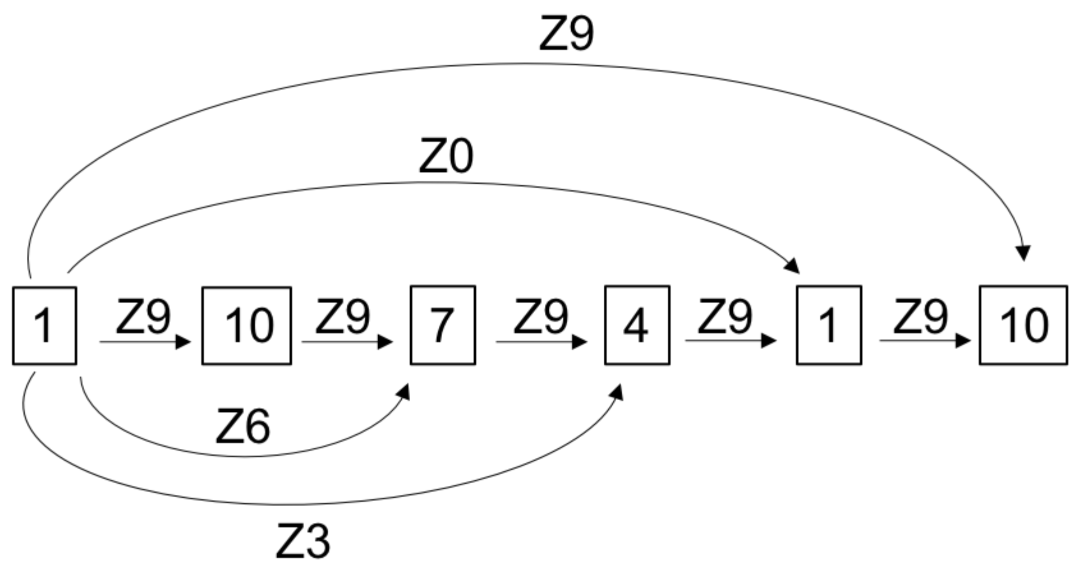
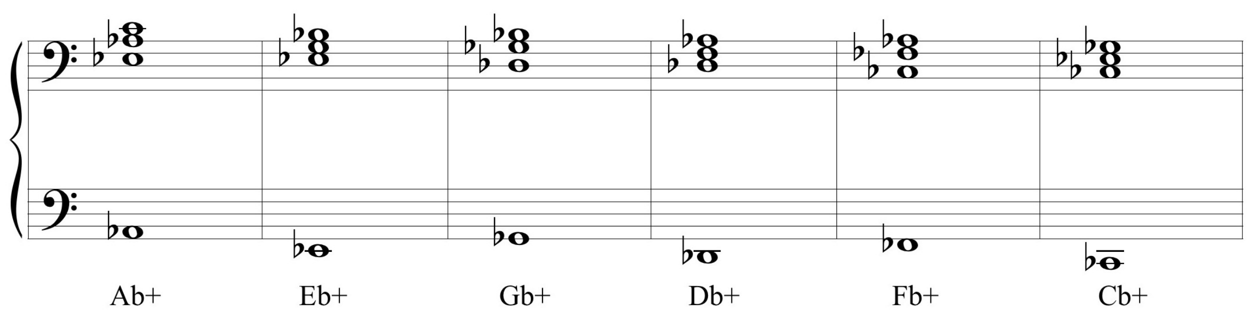


Figure 2. Progression 1 labeled in terms of sum classes and sum-class transformations

This concept might also be used prescriptively to determine how long a progression generated by a single sum-class transformation must continue in order to bring us to a chord in a specific relation with the chord with which the progression began. Any Z3 or Z9 progression must go through four iterations to reach a chord that is Z0 related to (belonging to the same sum class as) the first chord of the progression. Whether or not this last chord is actually the same as the first depends on what sort of transposition is involved. A series of T_7 transpositions (Z9), for example, will reach a chord that is Z0 related to the first after four iterations but will not reach the first chord again until twelve iterations have been completed. A progression by T_3 (also Z9), however, *does* reach the first chord after four iterations. The Z0 and Z6 transformations behave somewhat differently because these are the identity and halfway point of the sum-class space respectively. Composing either of these functions with itself results in chords that are either Z0 or Z6 related. Z3 and Z9 cannot be produced by repeated iterations of Z0 or Z6 (Definition 7 shows this to be true).

Let us turn now to Example 3, which reproduces a harmonic progression (“progression 2”) found shortly after progression 1.



Example 3. Harmonic reduction of mm. 122–128, “progression 2”

Progression 2 also consists of modally-matched triads, but this time, the triads are major. As can be seen in Figure 4, progression 2 results from the same series of transpositions seen in Figure 1. Utilizing the Z*n* group of transformations (see Figure 5) allows us to show the equivalence of these progressions from a voice-leading perspective despite the fact that they contain triads of different quality; something the Y*n*/X*n* group alone would not allow us to do.

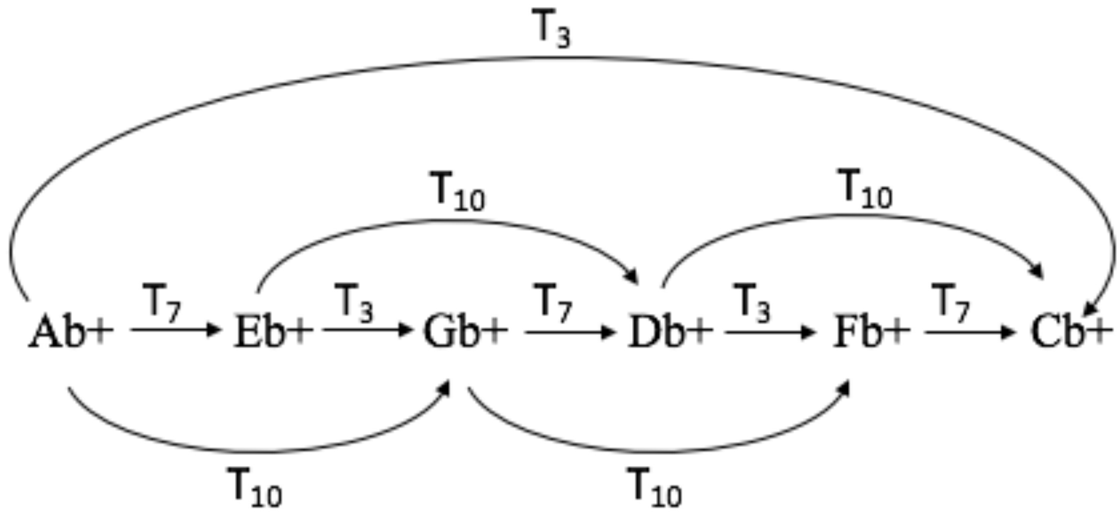


Figure 4. The transpositional transformations relating each chord of progression 2

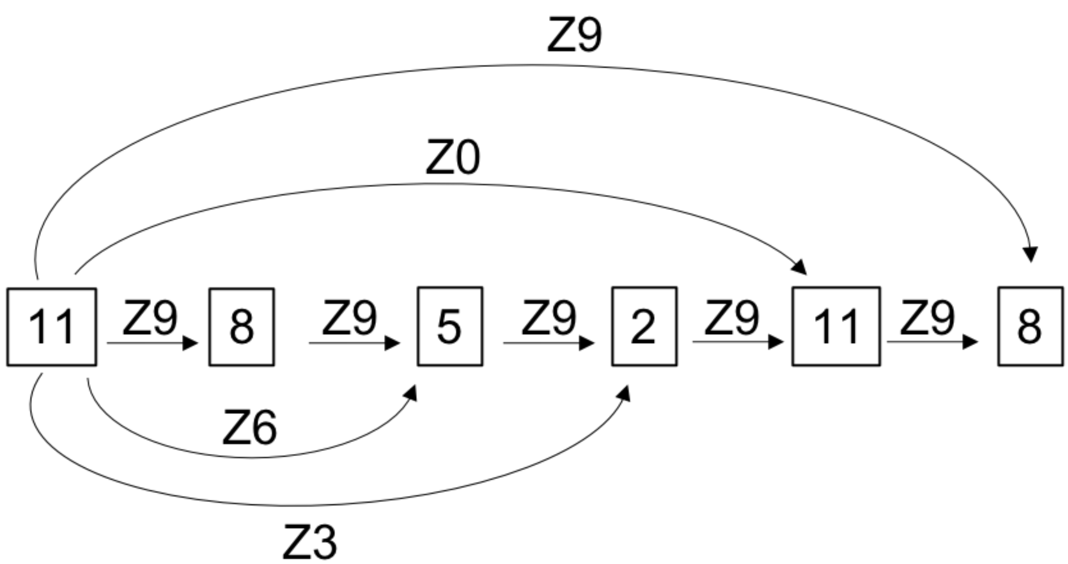


Figure 5. Progression 2 labeled in terms of sum classes and sum-class transformations

The sameness of the two progressions in sum-class (and thus voice-leading) space can also be seen in Figure 6. Both progression proceed around the figure in a counter-clockwise direction, always skipping over one sum class. Progression 2 is the same as progression 1 but rotated one click counter-clockwise. All that changes between the two progressions is the thus mode and root of each triad. In fact, the relationship between these two progressions is quite systematic: every triad in progression 2 shares its third with the triad in the same order position of progression 1. This relationship is known as the "slide" transformation, which is contained within X10 (see in Table 5). Because the Y*n*/X*n* and Z*n*/W*n* groups commute with one another, these relationships within and between the two progressions may be shown via the commutative transformational network seen in Figure 7. The X transformations are represented by double-sided arrows since any inversion (X transformation) is its own inverse. Slide (X10) thus takes A- (a member of 1) to Ab+ (a member of 11) and vice versa.

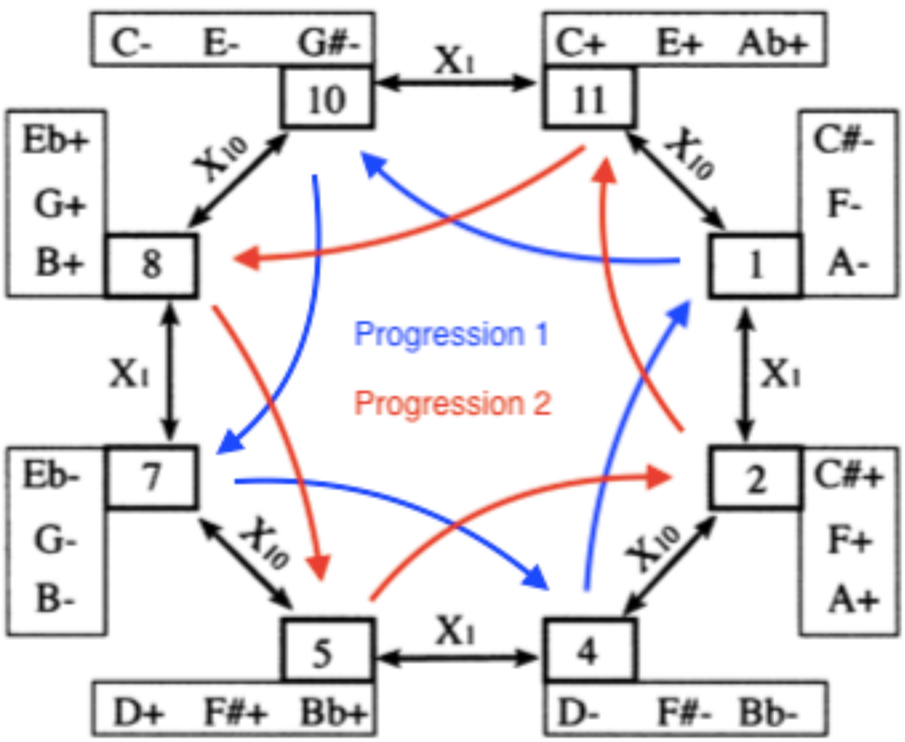


Figure 6. The motion of progressions 1 and 2 through the sum-class space[26]

[26] This figure utilizes Figure 3 from Cohn, "Square Dances with Cubes," 287.

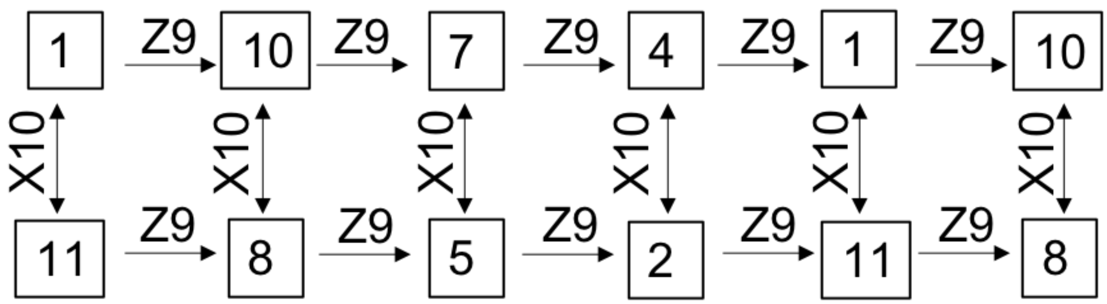


Figure 7. A commutative transformational network relating progressions 1 and 2

Progressions 1 and 2 serve important dramatic functions in the narrative of the song. The poem tells the story of a knight who is found "alone and palely loitering." When asked the reason for his current state, the knight tells a story (beginning in measure 36) of his meeting a woman "in the meads" with whom he seems to fall instantly in love. He eventually finds himself at her home where he is lulled to sleep. While he sleeps, the knight dreams a strange and terrible dream in which he sees "pale kings and princes" who inform him that he has been seduced by "La belle Dame sans merci." Progression 1 (m. 98) begins the knight's account of this dream. The cycle of consistent Z9 voice leading ends at measure 109 with the declaration of the ghostly figures that *La belle Dame* "hath thee in thrall!" Z9 voice leading begins again in measure 122 as the knight tells of his sudden awakening from the dream and continues until measure 130 when the knight returns once again to "reality."

Example 3 presented only an excerpt of progression 2 so that the relationship between it and progression 1 could be seen more clearly. The full progression as it appears in measures 122–130 actually includes an extra D+ and F- (see Figure 8). As can be seen, the succession of Z9 transformations extends for one more chord, producing a chord (D+), which is related to Ab+

by Z6. After this point, however, the Z9 cycle breaks when we are wrenched to F- via PRP (X4). As the song as a whole is in F minor, the introduction of F- at this point marks the end of the knight's dream world and the harsh "awakening" to the F minor "reality" of the rest of the song.

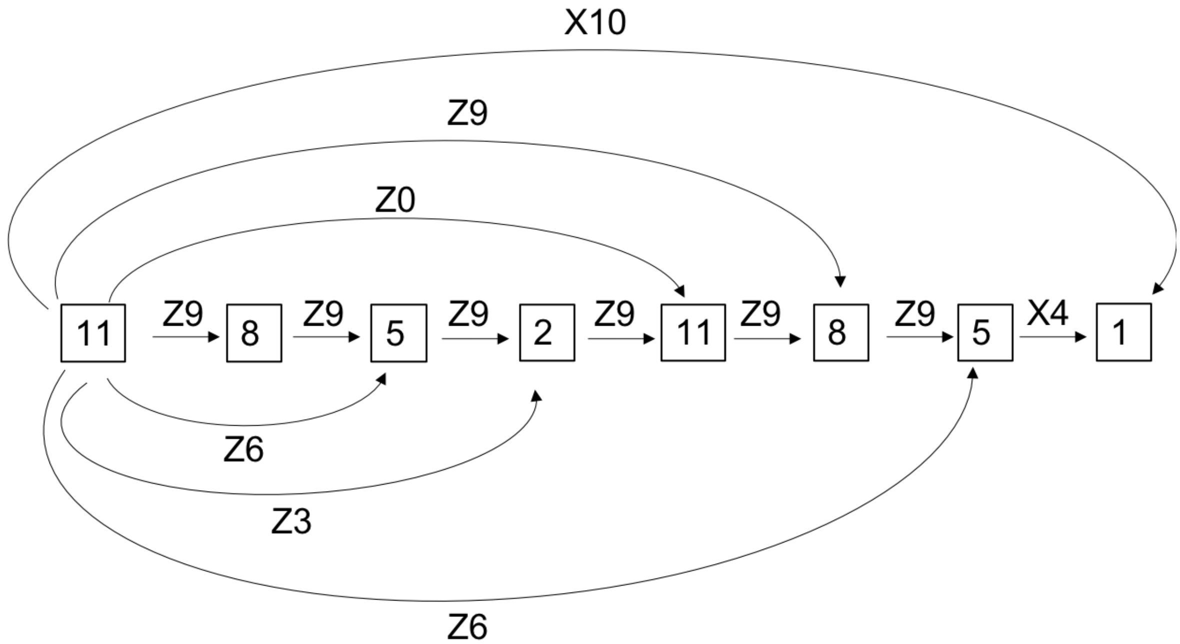


Figure 8. The full progression 2, mm. 122–130

F- also stands in a significant relationship with the two progressions we have studied. Ab+ (the chord that begins progression 2) and F- (the chord that ends progression 2) are related to one another by R, a transformation contained within X10, which is the same relationship we noted between each chord of progressions 1 and 2 in Figure 7. We can thus say that Ab+ is related to F- (the chord that breaks the consistent Z9 voice-leading cycle) in the same way that it is related to A- (the first chord of progression 1). Significantly, A- also stands in an R (X10) relationship with the C+ triad that breaks progression 1's Z9 cycle in measure 109. The opening and closing chords of each progression thus stand in the same relationship (R, contained in X10). Furthermore, the C+ chord that ends progression 1 in measure 109 belongs to the same sum class

(11) as the Ab+ which begins progression 2 in measure 122. The measures between these two chords, which are much less consistent from a voice-leading perspective, put the sequential voice leading of progressions 1 and 2 "on hold." Because this passage is bookended by the same sum class, I argue that they might be thought of as a prolongation of 11.

That the end of progression 1 and the beginning of progression 2 belong to the same sum class also suggests that these two progressions might be connected together into one large progression. Recalling that sum-class transformations group triadic transformations into equivalence classes based upon the distance they cover in directed voice-leading space, we can see that progressions 1 and 2 cover the same voice-leading distance since they are both generated by Z9. Indeed, referring to Figure 9, we can see that both progressions produce a consistent descent by whole step in the soprano and tenor voices and by half step in the alto. It should be noted, once again, that the voicing of the C- chord at the end of progression 1 represents the implied continuation of the voice-leading pattern. The chord actually contains Eb in the highest voice in measure 108. The implied voicing shown in Figure 9 nicely demonstrates the connection between the two progressions: after the interruption of the C+ chord and the prolongation of 11, the soprano and tenor pick up where they left off and continue on their whole-step descent. The F- chord that ends the progression stops this progress just short of completion. Had the progression been allowed to continue for one more iteration, it would have completed the whole-tone scales in the soprano and tenor (the two unique whole-tone scales) and the chromatic scale in the alto. It would also have brought us back to an A rooted chord. The progression could thus have started all over again. The F- wrenches us free of this never-ending cycle of hypnotic voice leading and returns us to the world of tonal harmony much like being wrenched free from a never-ending dream.

Figure 9. The combination of progressions 1 and 2, analyzed below the staff as triads and triadic transformations and as sum classes and sum-class transformations above the staff

Bibliography

Callender, Clifton, Ian Quinn, and Dmitri Tymoczko. "Generalized Voice-Leading Spaces." *Science* 320 (2008): 346–8.

Capuzzo, Guy. "Neo-Riemannian Theory and the Analysis of Pop-Rock Music." *Music Theory Spectrum* 26, no. 2 (2004): 177–200.

Clampitt, David. "Alternative Interpretations of Some Measures from *Parsifal*." *Journal of Music Theory* 42, no. 2 (1998): 321–34.

Clough, John. "A Rudimentary Geometric Model for Contextual Transposition and Inversion." *Journal of Music Theory* 42, no. 2 (1998): 297–306.

Cohn, Richard. "As Wonderful as Star Clusters: Instruments for Gazing at Tonality in Schubert." *19th-Century Music* 22, no. 3 (1999): 213–32.

_______. *Audacious Euphony*. New York: Oxford University Press, 2012.

_______. "Introduction to Neo-Riemannian Theory: A Survey and Historical Perspective." *Journal of Music Theory* 42, no. 2 (1998): 167–80.

_______. "Maximally Smooth Cycles, Hexatonic Systems, and the Analysis of Late-Romantic Triadic Progressions." *Music Analysis* 15, no. 1 (1996): 9–40.

_______. "Neo-Riemannian Operations, Parsimonious Trichords, and Their *Tonnetz* Representations." *Journal of Music Theory* 41, no. 1 (1997): 1–66.

_______. "Square Dances with Cubes." *Journal of Music Theory* 42, no. 2 (1998): 283–96.

_______. "Weitzmann's Regions, My Cycles, and Douthett's Dancing Cubes." *Music Theory Spectrum* 22, no. 1 (2000): 89–103.

Cook, Robert. "Transformational Approaches to Romantic Harmony and the Late Works of César Franck." Ph.D. diss., University of Chicago, 2001.

Douthett, Jack, and Peter Steinbach. "Parsimonious Graphs: A Study in Parsimony, Contextual Transformations, and Modes of Limited Transpositions." *Journal of Music Theory* 42, no. 2 (1998): 241–63.

Fiore, Thomas M., and Ramon Satyendra. "Generalized Contextual Groups." *Music Theory Online* 11, no. 3 (2005).

Hook, Julian. "Cross-Type Transformation and the Path Consistency Condition." *Music Theory Spectrum* 29, no. 1 (2009): 1–40.

_______. "Uniform Triadic Transformations." *Journal of Music Theory* 46, no. 1/2 (2002): 57–126.

Kramer, Jonathan D. "New Temporalities in Music." *Critical Inquiry* 7, no. 3 (1981): 539–56.

Kochavi, Jonathan. "Some Structural Features of Contextually-Defined Inversion Operators." *Journal of Music Theory* 42, no. 2 (1998): 307–20.

Klumpenhouwer, Henry. "Some Remarks on the Use of Riemann Transformations." *Music Theory Online,* 0, no. 9 (1994).

Lambert, Philip. "On Contextual Transformations." *Perspectives of New Music* 38, no. 1 (2000); 45–76.

Lewin, David. "Amfortas's Prayer to Titurel and the Role of D in *Parsifal*: The Tonal Spaces of the Drama and the Enharmonic Cb/B." *19th–Century Music* 7, no. 3 (1984): 336–49.

_______. *Generalized Musical Intervals and Transformations.* Oxford: Oxford University Press, 2011.

Morris, Robert D. "Voice-Leading Spaces." *Music Theory Spectrum* 20, no. 2 (1998): 175–208.

Ramirez, Miguel. "Chromatic-Third Relations in the Music of Anton Bruckner: A Neo-Riemannian Perspective." *Music Analysis* 32, no. 2 (2013): 155–209.

Rings, Steven. *Tonality and Transformation*. New York: Oxford University Press, 2011.

Satyendra, Ramon. "An Informal Introduction to Some Formal Concepts from Lewin's Transformational Theory." *Journal of Music Theory* 48, no. 1 (2004): 99–141.

Tymoczko, Dmitri. *A Geometry of Music: Harmony and Counterpoint in the Extended Common Practice.* New York: Oxford University Press, 2011.

_______. Generalizing Musical Intervals. *Journal of Music Theory* 53, no. 2 (2009): 227–54.

_______. "Scale Theory, Serial Theory and Voice Leading." *Music Analysis* 27, no. 1 (2008): 1–49.