Generalized Transformational Voice-Leading Systems

Thesis

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By

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Abstract

David Lewin writes that, "In conceptualizing a particular musical space, it often happens that we conceptualize along with it, as one of its characteristic textural features, a family of directed measurements, distances, or motions of some sort. Contemplating elements s and t of such a musical space, we are characteristically aware of the particular directed measurement, distance, or motion that proceeds 'from s to t.'"[1] This thesis is concerned with these measurements, distances, and motions as they relate to the voice leading between two pitch-class sets. We begin with Richard Cohn's idea that we might understand the total voice-leading interval between two pitch-class sets as the mod-12 sum of the pitch-class intervals traversed by each voice. This "pairwise voice-leading sum" (PVLS) allows us to see that the total voice-leading interval is the same between several pitch-class sets within the same T_n/I_n set class and also that several pitch-class set transformations will produce the same voice-leading interval when applied to any one set. These sets that are equidistant from a given point are grouped into equivalence classes known as "SUM classes" (because all such sets will return the same value when their constituent pitch classes are summed together mod 12) and the transformations producing the same voice-leading intervals are grouped into equivalence classes known as "SUM-class transformations." When the set class is not inversionally symmetrical, these

[1] David Lewin, *Generalized Musical Intervals and Transformations* (Oxford: Oxford University Press, 2011): 16.

transformations will be non-commutative, and we will be able to define a “dual” group of transformations for both the SUM classes and pitch-class sets that provide us with two different ways to navigate through the spaces. Together, the SUM classes/SUM-class transformations and pitch-class sets/pitch-class set transformations form two interrelated Generalized Interval Systems that allow us to conceptualize the “measurements, distances, and motions” of any of the T_n/I_n set classes and even, in a modified form, for all of the pitch-class sets of the same cardinality. What these constructions reveal, above all, is just how similar the set classes of the same cardinality really are as well as how many different ways there are to express the same background voice leading structures.

Dedication

To my wife, Heather.

Acknowledgments

I wish to begin by acknowledging my wife for her patience with me during these past months while this document completely consumed all the time that was rightfully hers. Not only was she always supportive, but she was also a voice of reason on those days where I felt like giving up and believed in me when I didn't believe in myself. She even stayed up with me until the wee hours of the morning the night before I submitted to my committee to help me edit and format this thesis.

I wish thank Dr. David Clampitt for his seminar in transformational music theory that first started me down the path that led to this thesis and for all of the ways that he has helped it (and me) to grow along the way. He was the one to suggest that group theory might provide a meaningful perspective for this project and was an invaluable resource as I tried to come to terms with the mathematical formalism this perspective required. He was also very kind to always respond promptly to the myriad (often rather lengthy) emails that I bombarded him with—even when he was on leave. Above all, his passion for this project and the promise he saw in it were a constant source of encouragement for me.

I wish to thank the entire faculty and staff of the School of Music at Southern Adventist University for the formative role they played in my life. These wonderful people not only taught me how to be a better musician and student but also a better person. I particularly want to acknowledge Judy Glass and Drs. W. Scott Ball, Peter Cooper, Gennevieve Brown-Kibble for their faith in me and for all of the opportunities they provided me during my time there.

I wish to thank Carol Christofferson for her music theory class at Pima Community College that first ignited my passion for music and music theory. It is no exaggeration to say that this class changed my life, and I will be forever indebted to Carol and her love of music theory.

I wish to thank my parents for raising me to be the person that I am today and for being supportive of my dreams even when they led down uncertain paths. I want to thank my mother for the opportunities afforded to me because of her choice to homeschool me and for bugging me to enroll at Pima Community College in the first place. I want to thank my father for all his years of hard work that made it possible for me to go to college.

Finally, and most importantly, I wish to thank God for all of the prayers He has listened to and answered in regard to this thesis. The fact that this document exists at all is a testament to the power of prayer. He has been so patient with my doubts and has continued to bless me richly despite them. I owe every success to Him and to Him alone. *Soli Deo gloria.*

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Chapter 1: Introduction to the SUM-Class System

Voice leading, as the name implies takes place at the level of the individual "voice." In most multipart contexts there is a one-to-one relationship between voices and parts. For example, each of the four parts in a string quartet constitutes its own voice. It may also be possible for a single part to imply or actually articulate what we would consider to be several different voices. Such is the case in Example 1.1, where the large registral gaps in the figuration help to separate the single line into the three separate voices seen in Example 1.2.[2] Conversely, the single melodic line we hear at the opening of the fourth movement of Tchaikovsky's 6th Symphony (Example 1.3) is not actually played by any one section of the orchestra, but is instead produced by the combination of the first and second violin lines seen in Example 1.4. Thus, we can see that there is not always a one-to-one correspondence between the number of parts in a musical texture and the number of voices we hear.

In any case, when speaking of the voice leading between two sonorities, we may say that a voice is any *pair* of pitches (x and y) between which a connection is suggested by some aspect of the music (register, instrumentation, timbre, articulation, duration, etc.).[3] That is, there is

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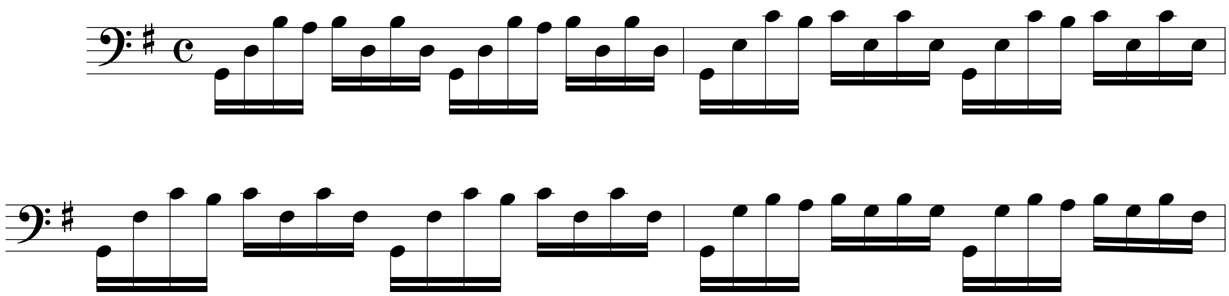
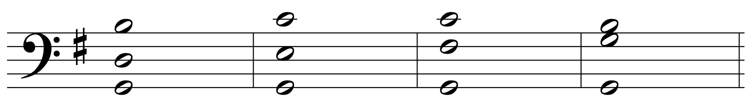
[2] For a discussion of this phenomenon, see David Huron, *Voice Leading: The Science behind a Musical Art* (Cambridge: The MIT Press, 2016), 63–86.

[3] See Joseph N. Straus, "Total Voice Leading," *Music Theory Online* 20, no. 2 (2014), http://www.mtosmt.org/issues/mto.14.20.2/mto.14.20.2.straus.html.

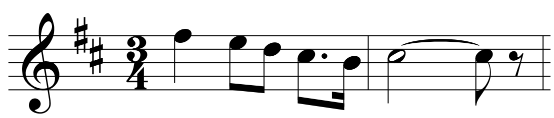
something within the music that implies that pitch *x* in the first sonority moves to pitch *y* in the second sonority, rather than to pitch *z*.



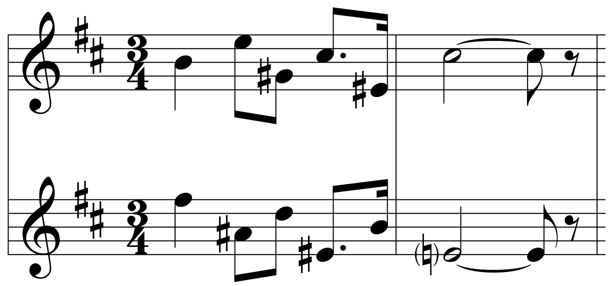
Example 1.1. J. S. Bach, Suite for Solo Cello in G Major, BWV 1007, mm. 1–4.



Example 1.2. A harmonic reduction of J. S. Bach, Suite for Solo Cello in G Major, BWV 1007, mm. 1–4.

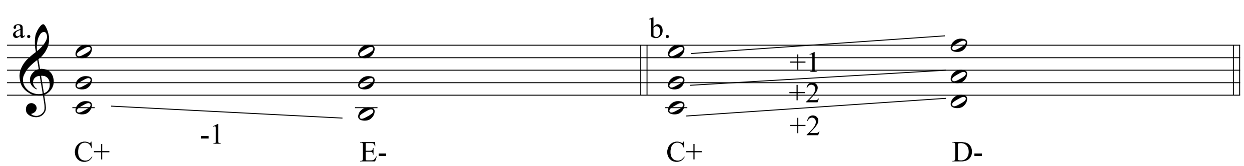


Example 1.3. The violin melody we hear in Tchaikovsky's Symphony No. 6, Op. 74, MMT IV, mm. 1–2.



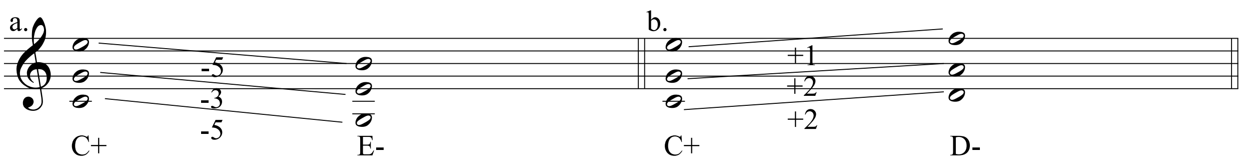
Example 1.4. What the violins actually play in Tchaikovsky's Symphony No. 6, Op. 74, MMT IV, mm. 1–2.

To speak of the voice leading between whole sonorities, then, is to speak *collectively* of the intervals traversed by each individual voice. The less each voice moves, the more efficient, smooth, or parsimonious the voice leading is said to be. Thus, we would say that the voice leading from the C-major triad (notated as C+ from now on) to the E-minor triad (notated as E- from now on) in Example 1.5 is more efficient than the voice leading from C+ to D- in Example 1.5 because the interval traversed by each voice in from C+ to E- is smaller than that from C+ to D-. We can also conceptualize this same notion in terms of the "voice-leading interval" between two sonorities. In this way, two sonorities can be said to be comparatively "closer" to one another in voice-leading space than two sets that require comparatively more motion on the voice-to-voice level. In this sense, we can say that the specific C+ and E- seen in Example 1.5 lie "closer" to one another than the C+ and D- in Example 1.5.



Example 1.5. The most efficient voice leading from C+ to E- (a) and from C+ to D- (b) as measured in the number of semitones traversed by each voice.

However, we cannot say that C+ and E- are always closer to one another in general than C+ and D-, because it is also possible to voice these triads such that the converse would actually be true, as can be seen in Example 1.6. This is because we have been discussing voice leading within the context of *pitch space* up to this point, and within pitch space, voices are always fixed at specific registral positions. We can thus say that C4 is closer to D4 than to E4, but this does not mean that *any* D would be closer to any C than any E. In order to make these kinds of generalizations, we will need to move from the realm of pitch space to the more abstract realm of pitch-*class* space.



Example 1.6. A situation in which C+ to E- (a) is further than from C+ to D- (b).

A pitch class is the set of all pitches that can be reduced to the same fundamental frequency (give or take some mistuning) once octaves are factored out. Thus, C4, C5, B#2, and Dbb7, and any other octave or enharmonic equivalent (even those extending infinitely beyond the range of human hearing) all belong to the same pitch class because they are all some power-of-two multiple of the same fundamental frequency. These pitch classes are an example of what are known as "equivalence classes," which require that any elements in the same class will all be "congruent" to one another (under a carefully-defined notion of congruence) but not to any member of another class. These classes are defined formally as a relation (*R*) on a set of elements (*S*) such that:

1) for all elements of *S* ($s \in S$), the pair (*s*, *s*) belongs to *R*.
2) if the ordered pair of elements (*s*, *t*) is in *R*, then the ordered pair (*t*, *s*) is also in *R*.
3) if (*s*, *t*) is in *R* and (*t*, *u*) is in *R*, then (*s*, *u*) will also be in *R*.[4]

The most familiar example of an equivalence relation is modular arithmetic, which is an equivalence relation on all integers. In modulo-twelve arithmetic (which we shall spend considerable time with in this thesis), any two integers are considered congruent if their difference is a whole-number multiple of the modulus. Thus, 1 and 25 are congruent mod 12 because $25 - 1 = 24$ and $24 = 2 \times 12$. In terms of the formal definition above, we can see that any integer will be congruent to itself mod 12 and that the order in which these two integers are compared with one another does not matter. Furthermore, if the difference between two integers *a* and *b* is a whole-number multiple of twelve and the difference between *b* and *c* is also a whole-number multiple of twelve, it will always be true that the difference between *a* and *c* will be a whole-number multiple of twelve. For example, $37 - 25 = 12 = 1 \times 12$ and $25 - 1 = 24 = 2 \times 12$, then $37 - 1 = 36 = 3 \times 12$.

It should be easy to see that the pitch classes also fulfill these requirements: 1) any pitch will reduce to the same fundamental frequency as itself, 2) the order in which two pitches are compared to one another does not affect the result, and 3) if a given pitch (say, C5) is an octave multiple of a second pitch (C4) and a third pitch (C6) is an octave multiple of the first (C5), then

[4] Adapted from I. N. Herstein, *Topics in Algebra* (Waltham: Blaisdell, 1964), 6.

the first and third pitches (C6 and C4) will also be octave multiples. These pitch classes are typically assigned a number from 0 to 11 such that *any* C/B#/Dbb = 0, *any* C#/Db = 1, etc.

Intervals between these pitch classes are vastly different from intervals between registrally-defined pitches because there is no longer any way to measure the *direction* of the interval. Without direction, there is no longer a distinction between an ascending major second and a descending minor seventh, and any compound of these intervals. As such, the interval between pitch classes a and b is simply measured as the difference $b - a$ modulo 12, which can be thought of as the number of "hours" one must move clockwise around Figure 1.1 to get from a to b. Because pitch-class intervals are not directed, we must be mindful of the fact that intervals receiving large numbers might actually be small *descending* intervals at the pitch level. For example, the semitone descent from C4 to B3 in the voice leading from C+ to E- in Example 1.5 is a pitch-class interval of 11, which is clearly a smaller interval than the ascending whole step from C4 to D4 in the voice leading from C+ to D- in Example 1.5 despite the fact that this is a pitch-class interval of 2. Then, when comparing the size of pitch-class intervals, it will likely be more meaningful to think of "interval classes"—which consider the pitch-class intervals produced by an ascending and descending pitch interval to be congruent—than pitch-class intervals. Interval classes (abbreviated as IC from now on) are represented as the smaller of the values $a - b$ or $b - a$. Thus, pitch-class intervals 1 and 11 are both IC1, 2 and 10 IC2, 3 and 9 IC3, etc.

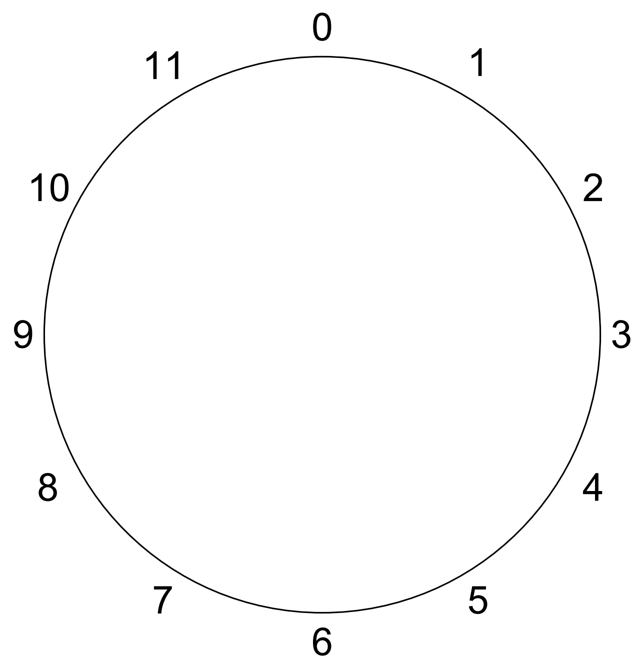


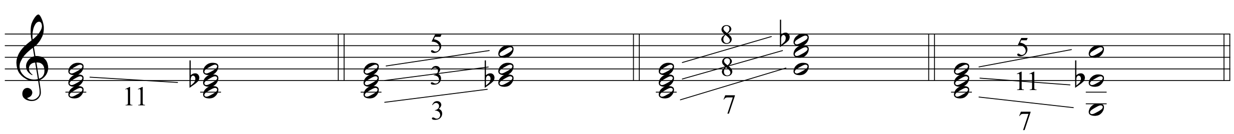
Figure 1.1. The pitch-class "clock."

With intervals measured in this way, we can then generalize the voice-leading interval between any two pitch-class sets (registrally-abstract sonorities) as the modulo-12 sum (that is, the total value reduced to within an octave) of semitones traversed by each voice. Because we are measuring the voice leading between ordered *pairs* of pitch classes that belong to the same voice (one in the first sonority and one in the second), we will refer to this as the pairwise voice-leading sum or PVLS,[5] which, as a function, is defined formally as:

[5] Not to be confused with Santa's *Parsimonious* Voice-Leading Sum of the same acronym. See Matthew Santa, "Nonatonic Systems and the Parsimonious Interpretation of Dominant-Tonic Progressions," *Theory and Practice* 28 (2003): 5. Santa's function takes the signed difference between pitch classes at the same order positions and then returns a sum of the absolute values of these differences. Thus, Santa's PVLS(C+, C-) = |(0 – 0)| + |(3 – 4)| + |(7 – 7)| = 0 + 1 + 0 = 1, whereas in Cohn's function (which I am calling PVLS) PVLS(C+, C-) = (0 – 0) + (3 – 4) + (7 – 7) = 0 + 11 + 0 = 11. Clearly these functions are very closely related, but Santa's function does not interact as meaningfully with the machinery we will be developing subsequently.

Definition 1.1. Let X and Y be pitch-class sets of cardinality z of the form $\{x_1, x_2, \ldots x_z\}$ and $\{y_1, y_2, \ldots y_z\}$ and let a pairwise voice-leading sum from X to Y (written as PVLS(X,Y)) = $\sum_{n=1}^{z}(y_n - x_n)$ modulo 12.[6]

A PVLS, then, is a simple subtraction problem in which each member of the first set is subtracted from the corresponding member of the second set. For example, PVLS(C+, D+) = ((2 – 0) + (6 – 4) + (9 – 7)) = (2 + 2 + 2) = 6. Interestingly, while the voice-leading interval created by each individual voice is affected by the ordering of the two sonorities, Example 1.7 reveals that the *total* voice-leading interval between the two sets is *not* affected by set order when the total is reduced to within an octave. This is because in a one-to-one mapping between two sets, the note "choices" available to each voice are directly tied to the note choices of the other voices. In terms of Example 1.7, when C moves to Eb instead of remaining on C, E must now move to either G or C and G must move to whatever is left over. As a result, any deviation from the most parsimonious voice leading in one voice will necessitate concomitant deviations in the other voices that will effectively cancel out under a PVLS.



0 + 11 + 0 = 11 3 + 3 + 5 = 11 7 + 8 + 8 = 11 (mod 12) 7 + 11 + 5 = 11 (mod 12)

Example 1.7. The PVLS from C+ to C- in various rotations.

[6] Adapted from Definition 4 in Richard Cohn, "Square Dances with Cubes," *Journal of Music Theory* 42, no. 2 (1998): 285; all equations will be modulo 12 from now on unless otherwise noted.

Because of this, Richard Cohn proves that we may dispense with the notion of voices altogether and define a PVLS in terms of an entire set by first summing their individual pitch classes together via the function SUM:

Definition 1.2. SUM(X) = $\sum_{n=1}^{z} x_n$ (mod 12), where z is the cardinality of X.[7]

Theorem 1.1. PVLS(X,Y) = SUM(Y) – SUM(X).

Proof: PVLS(X,Y) = $\sum_{n=1}^{z}(y_n - x_n)$ = $(y_1 - x_1)$ + $(y_2 - x_2)$ . . . + $(y_z - x_z)$ = $(y_1 + y_2 \ldots + y_z) - (x_1 + x_2 \ldots + x_z)$ = $\sum_{n=1}^{z} y_n - \sum_{n=1}^{z} x_n$ = SUM(Y) – SUM(X).[8]

Therefore, *as long as the two sets are of the same cardinality*, we may find their PVLS simply by subtracting the sum of the first set from the sum of the second set, meaning that the total voice-leading interval from a given set to any set of the same cardinality can easily be calculated by means of simple addition and subtraction. This also means that any two sets sharing the same SUM value will also lie the same PVLS interval from any given set, and, furthermore, because n – n will always equal 0, the PVLS between any two sets that have the same SUM value will always be 0.

This suggests that sets of the same cardinality could be placed into "equivalent" voice-leading classes according to their SUM values, with the result that a PVLS within a given class will always be zero and a PVLS between all sets of two given classes will always be the same. We can define this formally as a relation on any set of pitch-class sets as follows:

Definition 1.3. Let S be a set of pitch-class sets and R a relation on S such that $(s, t) \in R$ if and only if SUM(s) = SUM(t).

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[7] Adapted from Definition 5 in Cohn, "Square Dances," 286.

[8] Adapted from Theorem 1a and proof in Cohn, "Square Dances," 286.

Because this definition invokes the usual notion of equality, it is easy to see that it meets the criteria for an equivalence relation and will thus be left without formal proof. Each of these so-called "SUM classes" shall hereafter be notated in the form:

Definition 1.4. $\boxed{C}$ is the class of pitch-class sets s such that $s \in \boxed{C}$ if SUM(s) = C.[9]

Such a construction will allow us to map out the space of the pitch-class set universe and to measure intervals between individual "points" in terms of a PVLS. These voice-leading spaces can be constructed at several different organizational levels, and the following chapters will explore each in turn. Chapter 2 begins within the space of the twenty-four consonant triads of set class 3-11 for which this model was originally designed by Cohn, Chapter 3 will then explore what these SUM-class spaces look like within all of the other trichordal set classes and a small handful of set classes from each of the larger cardinalities, and Chapter 4 expands the SUM-class system to encompass entire cardinalities. Additionally, Appendix A examines a limited number of cases in which the SUM-class systems of two closely-related set classes can be united together into a single system, and Appendix B sketches a slight modification to the PVLS function that makes it possible to extend the ideas of Chapter 4 to pitch-class sets of different cardinality.

[9] Adapted from Definition 6 in Cohn, "Square Dances," 286.

Chapter 2: SUM Classes for the Consonant Triads

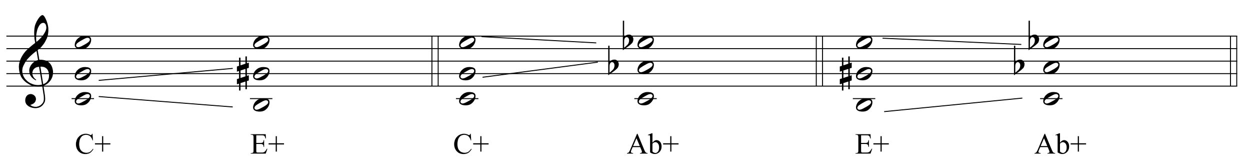
Applying the SUM function to the twelve major and twelve minor triads of set class 3-11 yields the partitioning into eight SUM classes seen in Table 2.1. Worthy of note is the symmetrical arrangement of the SUM classes themselves and the triads they contain. The eight SUM-class numbers take on the same values as the pitch classes in octatonic (1, 2), and each of these classes contains the three same-quality triads whose roots belong to the same augmented triad. The most interesting feature of these classes, however, is their relationship to voice leading. As noted earlier, the total voice-leading intervals among the three triads within each class is always zero, and the total voice-leading interval between the triads in different classes is simply the difference between their SUM classes. The neutrality of the voice leading within a single class results from the fact that voice leading between chords of the same quality whose roots lie a major third apart always involves a common tone in one voice and same-interval contrary motion between the other two voices (see Example 2.1). When these complementary intervals are summed together during the PVLS equation, they cancel one another out. Thus, while voice-leading motion is certainly happening within individual voices in these cases, there is no *net* voice leading at the level of entire pitch-class sets.[10] The relationship of this voice-leading consistency to the SUM classes is captured and generalized nicely by Cohn's eight "SUM-class transformations" (Y0, Y3, Y6, Y9, X1, X4, X7, X10) whose behavior is defined

[10] Cohn, "Square Dances," 285.

formally in Definition 2.1 (where TR*n* is a subscripted SUM-class transformation) and outlined in Table 2.2.

SUM Class	Triadic Members
1	{A-, F-, C#-}
2	{A+, F+, C#+}
4	{D-, F#-, Bb-}
5	{D+, F#+, Bb+}
7	{Eb-, G-, B-}
8	{Eb+, G+, B+}
10	{C-, E-, G#-}
11	{C+, E+, G#+}

Table 2.1. The eight SUM classes of set class 3-11.



Example 2.1. Contrary motion in the voice leading within 11.

Definition 2.1. TR*n*(*s*) = *s* + *n* (modulo 12) if *s* is congruent (≡) to 1 modulo 3; *s* – *n* (modulo 12) if *s* ≡ 2 modulo 3.[11]

[11] Originally Definition 7 in Cohn, "Square Dances," 288.

Y*n*/X*n* SUM-Class Transformations	**Action on SUM Classes**
Y0	(1) (2) (4) (5) (7) (8) (10) (11)
Y3	(1, 4, 7, 10) (2, 11, 8, 5)
Y6	(1, 7) (2, 8) (4, 10) (5, 11)
Y9	(1, 10, 7, 4) (2, 5, 8, 11)
X1	(1, 2) (4, 5) (7, 8) (10, 11)
X4	(1, 5) (2, 10) (4, 8) (7, 11)
X7	(1, 8) (2, 7) (4, 11) (5, 10)
X10	(1, 11) (2, 4) (5, 7) (8, 10)

Table 2.2. The permutations of the SUM classes achieved by the Y*n*/X*n* SUM-class transformations.[12]

The Y*n* transformations essentially act as the "transpositions" of the group because they map between classes containing triads of the same quality. That is, the Y-transform of any class containing major triads will also be a class containing major triads, and likewise for classes containing minor triads. But the Y*n* transformations differ from our usual notion of transposition in that they are defined to behave contextually (see Definition 2.1 above) so as to move in different "directions" when applied to classes containing minor triads (those congruent to 1 modulo 3) than when applied to classes containing major triads (those congruent to 2 modulo 3). Y3, for example, maps between classes containing minor triads that lie nine semitones away

[12] In this table and many subsequent like it, the actions of the SUM-class transformations are notated as cyclic permutations on a set of SUM classes. In this notation, the elements contained within each set of parentheses are mapped to one another in a cycle from left to right. Thus, the actions of Y3, for example, can be read as: 1 → 4 → 7 → 10 → 1 etc. and 2 → 11 → 8 → 5 → 2 etc. In the case of Y0, therefore, each SUM class is mapped to itself.

from one another, whereas it maps between classes containing major triads that lie three semitones from one another. The X*n* transformations are similarly contextual, but they act on the space like inversions—exchanging classes of opposite-quality triads and also acting as their own inverses.[13] By governing the movement from one SUM class to another, these transpositions essentially act as equivalence classes on total voice-leading intervals, which is reflected in the subscript of each transformation. Thus, each Y*n* and X*n* transformation moves between classes containing triads that lie *interval class n* semitones from one another other in terms of a PVLS.

From the actions of these transformations on the SUM classes seen in Table 2.2, we can derive the rule by which any two of these transformations combine with one another to create another transformation in the set. This is known as the "binary composition" or product of the elements of a set. Before discussing this, however, a note about the use of functional orthography in this thesis is in order. Thus far we have always written functions to the *left* of the element they act upon, like so: FUNC(x). This is known as left-functional orthography. When we combine or "compose" two functions, what we are essentially doing is applying one function first and then applying the second function to the result of the first. For example, to find the result of FUNC2(FUNC1(x)) we would first apply FUNC1 to x and then apply FUNC2 to the result of FUNC1 of x. In other words, we perform the right-most function first and then proceed from right to left. To abstractly compose two functions without reference to an element they act upon, then, we must still be sure to read and compose them from right to left. Thus, FUNC1 ∘ FUNC2 should be read as performing FUNC1 followed by FUNC2.

[13] Y6 also acts as its own inverse, but this is because, like the T_6 transformation on pitch-class space, 6 evenly divides 12 in half, meaning that adding or subtracting 6 (modulo 12) from the same number twice will always return the original number.

We may now define the binary composition of the SUM-class transformations as seen in Table 2.3. Reading this table from the left column (LC) to the top row (TR) results in the compound transformation LC(TR()), which, because we are using left functional orthography, means that it is the right-most transformation that would be performed first. In other words, the transformation in the top row is always performed first, followed by the transformation in the left column. As an example, X1(X4(1)) = X1(5) = 4, and the transformation that takes 1 to 4 in one move is Y3. Thus, we see it is true that X1 ∘ X4 = Y(4 – 1) or Y3, just as seen in the table. The full table of all possible pairings of these transformations (known as a Cayley table) is seen in Table 2.4.

	Y*m*	X*m*
Y*n*	Y(*m* + *n*)	X(*m* – *n*)
X*n*	X(*m* + *n*)	Y(*m* – *n*)

Table 2.3. The binary composition of the Y*n*/X*n* SUM-class transformations.[14]

[14] Adapted from Figure 5 in Cohn, "Square Dances," 289.

	Y0	Y3	Y6	Y9	X1	X4	X7	X10
Y0	Y0	Y3	Y6	Y9	X1	X4	X7	X10
Y3	Y3	Y6	Y9	Y0	X10	X1	X4	X7
Y6	Y6	Y9	Y0	Y3	X7	X10	X1	X4
Y9	Y9	Y0	Y3	Y6	X4	X7	X10	X1
X1	X1	X4	X7	X10	Y0	Y3	Y6	Y9
X4	X4	X7	X10	X1	Y9	Y0	Y3	Y6
X7	X7	X10	X1	X4	Y6	Y9	Y0	Y3
X10	X10	X1	X4	X7	Y3	Y6	Y9	Y0

Table 2.4. Cayley table for the composition of the Y*n*/X*n* SUM-class transformations.[15]

A quick test of any random compositions will reveal that the product of *any* two SUM-class transformations is also one of the eight SUM class transformations we have defined. In other words, there is no way to combine two transformations in the set and get a transformation not in the set. This is known as “closure” and is a property of an algebraic structure known as a group. A group is any set of elements (*S*) with a binary composition (∘) that fulfills each of the following criteria[16]:

1) If a and b are in S, then $a \circ b$ is also in S.
2) If $a, b, c \in S$, $(a \circ b) \circ c = a \circ (b \circ c)$.

[15] Also in Robert Cook, “Transformational Approaches to Romantic Harmony and the Late Works of César Franck” (PhD diss., University of Chicago, 2001), 101.

[16] Adapted from I. N. Herstein, *Topics in Algebra* (Waltham: Blaisdell, 1964), 26.

3) There exists an element $x \in S$ such that $x \circ a = a \circ x = a$ for all $a \in S$.

4) For every $a \in S$ there is an element a^{-1} such that $a \circ a^{-1} = a^{-1} \circ a = x$ (from 3 above).

Table 2.4 and a few examples will suffice to show that the SUM-class transformations are indeed a group. We have already seen that the SUM-class transformations fulfill the first criterion (closure), and the second column of Table 2.4 reveals that Y0 is the element (known as the identity) fulfilling the third criterion. This table also reveals that the inverse of every element in the set is also in the set, thus fulfilling the fourth criterion. The second criterion is slightly more involved, but as an example, we see that (X4 ∘ Y3) ∘ X7 = X7 ∘ X7 = Y0 and X4 ∘ (Y3 ∘ X7) = X4 ∘ X4 = Y0. This would also be true of any other combination of SUM-class transformations. In fact, if a set of elements under a given binary composition is able to create a structure like that seen in Table 2.4 where no two elements ever appear twice in a single row or column, then that set is a group.

When this group of eight SUM-class transformations (G) is made to act on the set of eight SUM classes (S), we can note that it will always be true that for any ordered pair of SUM classes that there will be one and only one SUM-class transformation that will send the first SUM class to the second. Formally, for any ordered pair $(s, t) \in S \times S$ (the Cartesian product of S, which is the set of all ordered pairs in S), there will be one and only one $g \in G$ that satisfies the equation $g(s) = t$. Groups that act upon a set in such a way are said to be "simply transitive."[17] In such cases, we can thus say that (s, t) uniquely determines g and define an interval function int that sends $S \times S$ to G via $\text{int}(s, t) = g$. The ordered triple (S, G, int) then satisfies Lewin's

[17] See David Lewin, *Generalized Musical Intervals and Transformations* (Oxford: Oxford University Press, 2011), 157.

definition of a Generalized Interval System (GIS) for the SUM classes.[18] As the name implies, a allows us to generalize all transformations and intervals within the space of the GIS.

As of yet, however, our GIS only allows us to generalize intervals between *classes* of major and minor triads but not between the triads themselves. Indeed, we have been very careful to note that the SUM-class transformations do not actually transform triads themselves but only the classes that contain them. In order to generalize intervals between specific triads, then, we must invoke a separate set of transformations that are defined to operate on triads. The so-called "neo-Riemannian" P, L, and R transformations are some examples of such transformations.[19] P and R transform triads by taking them to their parallel and relative triads respectively and L takes a major triad to its mediant and a minor triad to its submediant. Thus, P(D+) = D-, R(D+) = B-, and L(D+) = F#-. Importantly, these transformations are each their own inverses, meaning that we will end up back at D+ if we apply the same transformation to each of the products above: P(D-) = D+, R(B-) = D+, and L(F#-) = D+. In other words, these transformations are also contextual like the SUM-class transformations because they move in opposite "directions" depending upon the quality of triad they are applied to.

These transformations can also be composed together, but P, L, and R by themselves do not produce a closed group. For example, L(P(D+)) = L(D-) = Bb+, and Bb+ was not a triad that could be produced using only P, L, or R. Furthermore, note that LP takes a major triad to another major triad and would do likewise for a minor triad as well: L(P(D-)) = L(D+) = F#-. P, L, and R

[18] My thanks to David Clampitt for the wording of this paragraph. See Lewin, *Generalized Musical Intervals*, 26 and 157–9. For more on Generalized Interval Systems in general, see Ramon Satyendra, "An Informal Introduction to Some Formal Concepts from Lewin's Transformational Theory," *Journal of Music Theory* 48, no. 1 (2004): 99–141.

[19] For more on neo-Riemannian transformations, see Richard Cohn, "Introduction to Neo-Riemannian Theory: A Survey and a Historical Perspective," *Journal of Music Theory* 42, no. 2 (1998): 167–80.

thus act on triads like inversions because they *reverse* the quality of the triad, whereas PL acts like a transposition because it *preserves* quality. As we saw with the SUM-class transformations, composing two X*n* transformations (which we noted acted like inversions) always produced a Y*n* transformation (which we noted acted like transpositions). If we were to compose three neo-Riemannian transformations together (or, for that matter, three X*n* transformations), we would once again find quality to be reversed: R(L(P(D+))) = R(L(D-)) = R(Bb+) = G-. From this we can derive a general principle, namely that an odd number of inversions (1, 3, 5, etc.) is always another inversion but an even number of inversions (2, 4, 6, etc.) is always a transposition.

By composing P, L, and R in various ways, it is actually possible to produce twenty-four unique transformations (twelve transpositions and twelve inversions) that make it possible to transformationally navigate from any major or minor triad to any other major or minor triad (including itself): {E[20], P, L, R, PL, LP, RP, RL, LR, PR, PLP, PRP, LRP, LRL, RLR, RPR, LPR, PLR, RPLP, LRPR, RPRP, RPRL, PLPR, RPLPR}.[21] Though binary composition of these transformations certainly are not as intuitive as the SUM-class transformations (and this group is so large as to make a Cayley table very impractical), they are indeed closed under their binary composition.

As a random example, PLP(RPLPR(D+)) = PLP(C-) = E+ and D+ can also be sent to E+ via LRPR (also a transformation in the group). These transformations are also associative, which can easily be seen because (P ∘ L) ∘ R and P ∘ (L ∘ R) both produce PLR, and likewise for any

[20] This is the "identity" transformation that takes any triad to itself.

[21] It is actually possible to construct these transformations using only L and R transformations, see Satyendra, "An Informal Introduction," 118. I have chosen to make use of P as well since Cohn constructs the transformations in this way in "Square Dances."

other triplets. We have also seen that P, L, and R are their own inverses, and this will also be true of any of the other inversion transformations. For example, LRP(D+) = A- and LRP(A-) = D+. The inverse of any of the transpositions will just be the same transformations read backwards: LP(D+) = Bb+ and PL(Bb+) = D+. Finally, as the identity, the E transformation will compose with any transformation to produce that same transformation. Thus, the set of twenty-four unique neo-Riemannian transformations satisfy all four criteria for a group![22] Additionally, this group acts simply transitively on the set of twenty-four major and minor triads (the consonant triads from now on), meaning that we can also define a GIS for them just as we did for the SUM classes and SUM-class transformations above:

> **Definition 2.2.** Let S be the set of twenty-four consonant triads and G the group of twenty-four neo-Riemannian transformations. Because there is one and only one g in G that satisfies the equation $g(s) = t$ for all $s \in S$, we can then define an interval function int: $S \times S \rightarrow G$ via int(s, t) = g. Together, S, G, and int define a GIS.

Though certainly interesting on its own, the most significant feature of this GIS in the context of this thesis is its relationship to the GIS of the SUM classes and SUM-class transformations. To see this relationship, though, we must first investigate the behavior of small subsets of the triadic/neo-Riemannian GIS itself.

If within a group (G) there exists a subset (H) that also forms a group on its own, this subset is known as a "subgroup."[23] Subgroups may be of various sizes or "orders" and contain any transformations from the group, but Herstein shows that the order of the subgroup will always divide the order of the group and that any subgroup will always contain the group identity

[22] For further discussion of the group structure of the neo-Riemannian transformations, see Satyendra, "An Informal Introduction," 118–23.

[23] Herstein, *Topics in Algebra*, 32.

element.[24] Because the neo-Riemannian group is of order twenty-four, we know that any of its subgroups can only be of order one (just the identity element), two, three, four, six, eight, twelve, or twenty-four (the group itself). This significantly limits the number of possible subgroups that exist within the neo-Riemannian group, but there are still more than we have time to consider here. The subgroup that we will be interested is the order-three subgroup {E, PL, LP}. If we were to check this subgroup for all four group criteria, we would find that it does indeed meet the definition of a group. But this is actually unnecessary because Herstein proves that any nonempty finite subset that is closed under the group binary composition will form a subgroup.[25] Thus, since PL ∘ LP = E and any possible composition of E and another transformation will just produce that same transformation, this group is closed and so forms a subgroup.

Once we have defined a subgroup (H), we can then say that any two elements of the parent group ($f, g \in G$) are congruent to one another modulo the subgroup if the product of the first element with the *inverse* of the second is an element in the subgroup. Formally, $f \equiv g$ for $f, g \in G$ if $f \circ g^{-1} \in H$.[26] In the context of neo-Riemannian transformations, we can say that any two transformations are congruent to one another mod {E, PL, LP} if the product the first transformation and the inverse of the second is equal to E, PL, or LP. For example, LR and PR can be said to be congruent to one another mod {E, PL, LP} because LR ∘ RP (the inverse of PR) = LRRP = LP ∈ {E, PL, LP}. Herstein goes on to show that this congruence mod H is in fact

[24] Herstein, *Topics in Algebra*, 32.

[25] Herstein, *Topics in Algebra*, 33.

[26] Herstein, *Topics in Algebra*, 34.

an equivalence relation, meaning that LR and RP belong to the same mod-*H* equivalence class.[27] We can generate all of these mod-*H* equivalence classes by composing the three elements of *H* against all of the elements of its parent group ($g \in G$) in the order $h \circ g$. Doing so for the neo-Riemannian group results in the eight equivalence classes seen in Table 2.5, which are known as the left "cosets" (because we are using left-functional orthography) of *H* in *G*.[28]

{E, PL, LP}

{RP, RL, RPLP}

{LRPR, RPRP, RPRL}

{PLPR, LR, PR}

{P, L, PLP}

{PRP, LRP, LRL}

{RLR, RPR, RPLPR}

{R, LPR, PLR}

Table 2.5. The eight cosets of {E, PL, LP} in the neo-Riemannian group.

We could also generate all of the right cosets of *H* in *G* by composing the elements of *H* and *G* in the opposite order ($g \circ h$). In many cases, these left and right cosets will actually be different from one another, but the {E, PL, LP} subgroup possesses the distinguished quality of having identical left and right cosets. Subgroups of this kind are known as "normal" subgroups,

[27] Herstein, *Topics in Algebra*, 34.

[28] Herstein, *Topics in Algebra*, 34.

and Herstein shows that the product of any two cosets of a normal subgroup is also coset of that subgroup.[29] In other words, the set of cosets is closed under the group binary operation, and we saw earlier that any nonempty finite subset of a group that is closed under the group binary composition is itself a group (subgroup). Thus, the set of cosets of a normal subgroup *is also a group* and is known as the "quotient group" *G/N* such that *G* is the parent group and *N* the normal subgroup. What this means in the context of the neo-Riemannian group is that all of the equivalence classes modulo {E, PL, LP} form a group of their own!

All this talk about equivalence classes for transformations begs the question as to what is actually congruent about these transformations. Congruent transformations certainly do not produce the same mappings on triads (for example, LR(D+) = G+ and PR(D+) = B+) nor are they even necessarily inverses of one another (though some are, see PL and LP). But if we apply any three congruent transformations to the same triad, we can see that these transformations are the same in terms of the voice-leading interval between the triads they map: P(D+) = D-, L(D+) = F#-, PLP(D+) = Bb-, and PVLS(D+, D-), PVLS(D+, F#-), and PVLS(D+, Bb-) all equal 1! Furthermore, we also saw earlier than any three triads that lie the same PVLS interval from any other triad all belong to the same SUM class, which means that congruent neo-Riemannian transformations will always map between triads that belong to the same SUM classes. In fact, the actions of the eight cosets seen in Table 2.5 on the triads produce the *exact* same mappings at the level of the SUM classes as the SUM-class transformations! For example, P, L, and PLP will each map any triad in [1] to a triad in [2], any triad in [2] back to a triad in [1], a triad in [4] to a triad in [5], a triad in [5] to a triad in [4], a triad in [7] to a triad in [8], a triad in [8] to a triad in [7], a triad in [10] to

[29] Herstein, *Topics in Algebra*, 42–43.

a triad in $\boxed{11}$, and a triad in $\boxed{11}$ to a triad in $\boxed{10}$. In other words, the exact same mappings on the SUM classes as X1 (see Table 2.2).

We can thus define a mapping (*f*)—known as a "homomorphism"—from the neo-Riemannian group (*G*) onto the SUM-class transformation group (*G'*) such that the product of two neo-Riemannian transformations is sent to the product of two SUM-class transformations under the mapping.[30] Or, formally, for all $a, b \in G, f(a \circ b) = f(a) \circ f(b) \in G'$. The "kernel" of this homomorphism is the set of elements in *G* {E, PL, LP} that are sent to the identity element of *G'* (Y0).[31] In group theory it is demonstrated that all kernels are normal subgroups, and all normal subgroups are kernels. By virtue of this equivalence, every homomorphism from a group *G* onto another group *G'* gives rise to a quotient group *G*/*K* where *K* is the kernel of the homomorphism.[32] In Table 2.6, we can see that each of the twenty-four elements in *G* (the neo-Riemannian transformations) are mapped to one of the eight transformations in *G'* (the Y*n*/X*n* transformations). Moreover, as we have seen, this mapping is a homomorphism, whose kernel is {E, PL, LP}, and Table 2.6 shows that there is a one-to-one correspondence between the cosets of this kernel and the elements in *G'*. The cosets form the quotient group *G*/{E, PL, LP}, and the homomorphism that sends this quotient group one-to-one onto the Y*n*/X*n* group is known as an "isomorphism."[33]

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[30] Herstein, *Topics in Algebra*, 46.

[31] Herstein, *Topics in Algebra*, 47.

[32] Herstein, *Topics in Algebra*, 47–48.

[33] Herstein, *Topics in Algebra*, 49.

A quick test reveals that the product of any two neo-Riemannian transformations will indeed be the product of the two SUM-class transformations they are mapped to. For example, R ∘ L = RL which gets sent to Y3 under the homomorphism. The homomorphism also sends R to X10 and L to X1, the product of which is also Y3. In essence, then, the homomorphism defines an equivalence relation on the neo-Riemannian transformations such that the equivalence classes (the cosets of {E, PL, LP}) may be identified with the SUM-class transformations via the equivalence relation defined formally in Definition 2.3 (with no proof since it invokes the usual notion of equivalence). The cosets form the quotient group, and that group is isomorphic to the Y*n*/X*n* group.

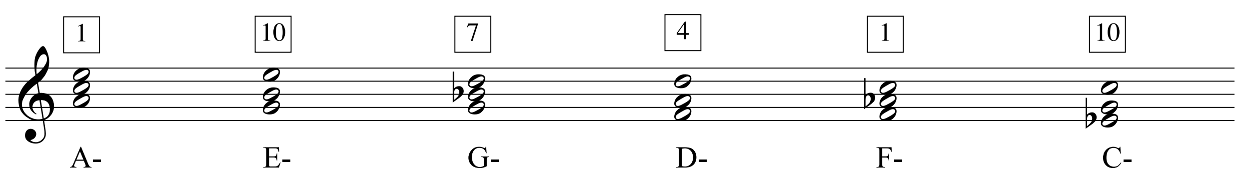
Neo-Riemannian Transformations	**Isomorphism**	**SUM-Class Transformations**
{E, PL, LP}	⇔	Y0
{RP, RL, RPLP}	⇔	Y3
{LRPR, RPRP, RPRL}	⇔	Y6
{PLPR, LR, PR}	⇔	Y9
{P, L, PLP}	⇔	X1
{PRP, LRP, LRL}	⇔	X4
{RLR, RPR, RPLPR}	⇔	X7
{R, LPR, PLR}	⇔	X10

Table 2.6. The homomorphism from the neo-Riemannian group onto the Y*n*/X*n* SUM-class transformation group mediated by the isomorphism between the quotient group of the neo-Riemannian group modulo {E, PL, LP} and the Y*n*/X*n* group.

Definition 2.3. Let S be the set of all twenty-four consonant triads, T the set of twenty-four neo-Riemannian transformations, and R a relation on T such that $s, t \in R$ for any $s, t \in T$ that satisfies the equation $\text{SUM}(s(a)) = \text{SUM}(t(a))$ for all $a \in S$.

What this homomorphism allows us to do, then, is to combine the GISs of the SUM classes/SUM-class transformations and consonant triads/neo-Riemannian transformations into one multilevel GIS, and by so doing, to generalize the voice-leading intervals between all consonant triads. In particular, this structure elucidates the ways in which many different surface-level triadic progressions are really just different realizations of the same background voice-leading structure.[34]

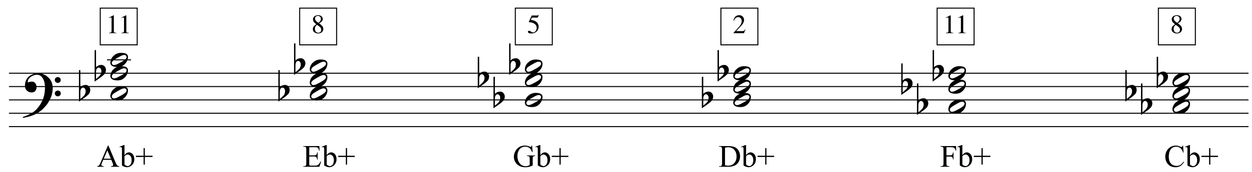
Powerful though this machinery may be, however, the contextuality of the neo-Riemannian and SUM-class transformations places limits on its analytical power. Consider, for example, two harmonic progressions from Charles Villiers Stanford's *La belle dame sans merci* seen in Examples 2.2 and 2.3.



Example 2.2. A harmonic reduction of Stanford, *La belle dame sans merci*, mm. 98–108.[35]

[34] See Cohn, "Square Dances" for examples of analyses using this construction.

[35] In the actual voicing of the C- chord in measure 108, the Eb is in the soprano and the G is omitted altogether. The voicing presented here is that implied by the continuation of the sequence.



Example 2.3. A harmonic reduction of Stanford, *La belle dame sans merci*, mm. 122–128.

In terms of a total voice-leading interval the two progressions are identical. In each, a single common tone is held while one of the remaining voices descends by a semitone and the other by a tone. The total voice-leading interval between the triads in both progressions, then, is nine semitones, which can also be seen easily by looking at the difference between the SUM classes to which each triad belongs in Examples 2.2 and 2.3. Yet while it is quite clear that the voice leading of the two progressions is the same, neither the SUM-class or neo-Riemannian transformations capture this sameness. In neo-Riemannian terms, the first progression is produced by an alternation of LR and PR while the second progression requires RL and RP. Likewise, Y9 connects the SUM classes in the first progression where the second uses a string of Y3 transformations. These are not simply different labels but, in fact, *opposite* ones. Indeed, we should not be surprised this is so, since both transformational groups are explicitly defined to act inversely upon major and minor triads.

The actions of such contextually-defined transformations make perfect sense in contexts where we are dealing with triads of opposite quality because it is easier to observe the same kinds of relationships in different contexts when we can say that the same P transformation can send C+ to C- and also send E+ to E-. Having to rely upon I_n transformations would make these relationships much more obscure since I_7 takes {0, 4, 7} to {0, 3, 7} whereas it is I_3 that takes {4, 8, 11} to {4, 7, 11}. Furthermore, we expect these inversional transformations to take us back *and forth* between a pair of triads, which of course necessitates that these transformations move in opposite directions when applied to triads of different quality. For transformational relationships between triads of the same quality, however, it is the T_n transformations that provide the most meaningful information because the same "interval" will receive the same T_n transformation whether it is between major or minor triads.

The T_n transformations cannot interact with the SUM-class transformations as they are currently defined, however, because they are *not* contextual. A string of T_1 transformations on major triads, for example, would progress in a cycle through $\boxed{2}$, $\boxed{5}$, $\boxed{8}$, and $\boxed{11}$ in that order, but there is no single Y*n* or X*n* transformation whose action corresponds. To make use of the T_n transformations will thus require the invocation of an altogether different group of SUM-class transformations. To begin looking for this new group, let us first consider some unusual features of the Y*n*/X*n* group.

We noted earlier that the Y*n* transformations acted like contextual transpositions and the X*n* like contextual inversions. The differences in the way these transformations behave causes them to combine with one another in rather unusual ways, as we saw in the Cayley table governing their binary composition (Table 2.4 above). This group structure is known as "non-commutative" or "non-abelian," and in such groups the order in which its members are composed affects the result.[36] Here, for example, X1(Y3($\boxed{1}$)) = $\boxed{5}$ whereas Y3(X1($\boxed{1}$)) = $\boxed{11}$. Group elements that behave in this way are said to be "non-commutative." Within the SUM-class transformations, only Y0 and Y6 commute with every other transformation: Y0 because it

[36] Herstein, *Topics in Algebra*, 27.

is the identity of the group and Y6 because of the special place that six holds within a mod-12 universe. Because of the homomorphism from the neo-Riemannian transformations onto the SUM-class transformations, any of the neo-Riemannian transformations sent to Y0 or Y6 will also preserve intervals at the level of the triad.

In a non-commutative GIS like this one, David Lewin has proved that not all transformations will preserve the interval between the two elements they are applied to.[37] We can observe this phenomenon in the Y3, Y9, X1, X4, and X10 transformations. For example, X1 sends $\boxed{1}$ to $\boxed{2}$, but if we transform $\boxed{1}$ and $\boxed{2}$ by X10 (or any of the other non-interval-preserving transformations) we retrieve $\boxed{11}$ and $\boxed{4}$, which are *not* related to one another by X10. In terms of a GIS, we can say that the interval between $\boxed{1}$ and $\boxed{2}$ is not the same as the interval between the X10-transform of $\boxed{1}$ and $\boxed{2}$. Formally, $\mathrm{int}(\boxed{1}, \boxed{2}) \neq \mathrm{int}(X10(\boxed{1}), X10(\boxed{2}))$. This also occurs at the level of the triads: int(A+, A-) = P, but int(R(A+), R(A-)) = int(F#-, C+) = RPR. The commutative transformations (Y0 and Y6 and the neo-Riemannian transformations sent onto them under the homomorphism), on the other hand will *always* preserve the interval between the two elements they are applied to: $\mathrm{int}(\boxed{4}, \boxed{8}) = X4$ and $\mathrm{int}(Y6(\boxed{4}), Y6(\boxed{8})) = \mathrm{int}(\boxed{10}, \boxed{2}) = X4$. Such transformations are known as "interval-preserving" transformations.[38]

Satyendra (via Lewin) shows that all non-commutative groups have a "dual" non-commutative group of transformations acting on the same space that will commute with every transformation in our first group.[39] We can find this commuting group through the algebraic

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[37] See Lewin, *Generalized Musical Intervals*, 48–50 and Satyendra, "An Informal Introduction," 131–34.

[38] Lewin, *Generalized Musical Intervals*, 48.

[39] See Satyendra, "An Informal Introduction," 131–34, and Lewin, *Generalized Musical Intervals*, 251–53.

method described in Satyendra, which involves manually computing the action of each new transformation by feeding it each SUM class in turn.[40] Doing so results in the set of eight transformations seen in Table 2.7.[41] The actions of these transformations are defined generally in Definitions 2.4 and 2.5 and the rule for their binary composition is summarized in Table 2.8.

Definition 2.4. Z$n(s) = s + n$.

Definition 2.5. W$n(s) = n - s$.

***Zn*/W*n* SUM-Class Transformations**	**Action on Sum Classes**
Z0	(1) (2) (4) (5) (7) (8) (10) (11)
Z3	(1, 4, 7, 10) (2, 5, 8, 11)
Z6	(1, 7) (2, 8) (4, 10) (5, 11)
Z9	(1, 10, 7, 4) (2, 11, 8, 5)
W0	(1, 11) (2, 10) (4, 8) (5, 7)
W3	(1, 2) (4, 11) (5, 10) (7, 8)
W6	(1, 5) (2, 4) (7, 11) (8, 10)
W9	(1, 8) (2, 7) (4, 5) (10, 11)

Table 2.7. The permutations of the SUM classes achieved by the Z*n*/W*n* SUM-class transformations.

[40] Satyendra, "An Informal Introduction," 131–34.

[41] This same group (though labeled differently) is also discussed in Cook, "Transformational Approaches to Romantic Harmony," 103–5. In fact, this chapter and Cook's second chapter are concerned with many of the same topics, but I only became aware of Cook's work after having already developed the ideas for this chapter. Though there are certainly similarities between our approaches, Cook appears mostly interested in the level of the SUM class and SUM-class transformation and does not make any mention of the relationship between the SUM-class transformations and the neo-Riemannian and T_n/I_n transformations.

	Zm	Wm
Zn	$Z(m + n)$	$W(m + n)$
Wn	$W(n - m)$	$Z(n - m)$

Table 2.8. The binary composition of the Z*n*/W*n* SUM-class transformations.[42]

We can easily see that this set is closed under the binary composition because adding or subtracting any combination of the integers 0, 3, 6, or 9 modulo 12 will always produce one of these same integers. From the mapping table (Table 2.7) it is apparent that Z0 is the group identity and that the inverse of each transformation is also in the set (every transformation except Z3 and Z9 is its own inverse, and these two transformations are each other's inverses). Finally, these transformations also associate one another such that, for example, (W3 ∘ Z3) ∘ W0 and W3 ∘ (Z3 ∘ W0) are both equal to Z0: (W3 ∘ Z3) ∘ W0 = W0 ∘ W0 = Z0 and W3 ∘ (Z3 ∘ W0) = W3 ∘ W3 = Z0. The action of this group on the SUM classes is also simply transitive, and so we can define a GIS with *S* as the SUM classes, *G* the Z*n*/W*n* group, and int a function that maps from pairs of elements in *S* to *G*. The only difference between this GIS and our earlier SUM-class GIS are the transformations contained in *G*. In other words, it is the same space but we merely navigate through it differently.

In addition to being the dual of the Y*n*/X*n* group, the Z*n*/W*n* group is isomorphic to the Y*n*/X*n* group because there is a homomorphism that maps these groups onto one another one-to-one. The mapping table for this isomorphism can be seen in Table 2.9. As a homomorphism, it will be true that the product of any two transformations from the Y*n*/W*n* group will be sent to the product of two transformations from the Z*n*/W*n* transformation under the isomorphism. For

[42] Adapted from Figure 5 in Cohn, "Square Dances," 289.

example, X7 ∘ Y3 = X10 and under the isomorphism X7 and Y3 are sent to W6 and Z3 whose product is W9—the image of X10 under the isomorphism. That these groups are isomorphic does not necessarily imply that they are identical or that they would act upon the SUM-classes in the same way (they do not), but it does guarantee that their internal group structures are the same. Thus, if the Y*n*/X*n* group is non-commutative, then the Z*n*/W*n* group (or any other group isomorphic to Y*n*/X*n*) will also be non-commutative. Though they are each non-commutative on their own, any pairing of one transformation from the Y*n*/X*n* group and one transformation from the Z*n*/W*n* group will commute with one another because these groups are each other's duals: Y3(W0($\boxed{1}$)) = Y3($\boxed{11}$) = 8 and W0(Y3($\boxed{1}$)) = W0($\boxed{4}$) = 8. This means that the Z*n*/W*n*-transform of two SUM classes related to one another by a particular Y*n*/X*n* transformation will still be related by the same Y*n*/X*n* transformation.

Y*n*/X*n* Transformations	**Isomorphism**	**Z*n*/W*n* Transformations**
Y0	⇔	Z0
Y3	⇔	Z3
Y6	⇔	Z6
Y9	⇔	Z9
X1	⇔	W0
X4	⇔	W3
X7	⇔	W6
X10	⇔	W9

Table 2.9. The isomorphism from the Y*n*/X*n* group to the Z*n*/W*n* group.

Though the Y*n*/X*n* and Z*n*/W*n* groups are isomorphic to one another, this does not necessarily imply that we can define a mapping directly from the neo-Riemannian group to the Z*n*/W*n* group in the same way we did to the Y*n*/X*n* group. Indeed, there is no single Z*n* or W*n* transformation that can even accommodate the actions of L, which always maps between triads in adjacent SUM classes like [1] and [2], [4] and [5], etc. Thus, in order to make a multilayered GIS with the Z*n*/W*n* transformations, we will need a different set of twenty-four triadic transformations whose quotient group modulo the kernel will be isomorphic to the Z*n*/W*n* group in a way analogous to the isomorphism from the neo-Riemannian group to the Y*n*/X*n* group.

We need not look far, then, for Satyendra has shown that the neo-Riemannian and usual transposition and inversion (T_n and I_n from now on) groups are not only isomorphic, but also each other's duals, which means that these two groups will also commute with one another.[43] For example, we saw earlier that int(A+, A-) = P while int(R(A+), R(A-)) = PRP, but if we were to transform these two triads by a T_n or I_n transformation instead of R, we would find that the transformational interval between them is preserved: int(I_0(A+), I_0(A-)) = int(Ab-, Ab+) = P.

But before we get ahead of ourselves, we ought to see that the twelve T_n and twelve I_n transformations together really do form a group. The rule for the binary composition for these transformations can be seen in Table 2.10. The group is closed under this binary composition because we are dealing with mod-12 arithmetic, and as such, adding or subtracting two mod-12 integers from each other will always produce another mod-12 integer. These transformations also associate with on another such that $(T_2 \circ I_1) \circ I_3$ and $T_2 \circ (I_1 \circ I_3)$ both equal T_0: $(T_2 \circ I_1) \circ I_3 = I_3 \circ I_3 = T_0$ and $T_2 \circ (I_1 \circ I_3) = T_2 \circ T_{10} = T_0$. Additionally, any I_n transformation will be its own

[43] See Satyendra, "An Informal Introduction," 118–23.

inverse and any T_n and T_{12-n} transformation will be each other's inverses. Finally, the product of T_0 with any other T_n or I_n transformation will be the later transformation, and so the twenty-four T_n/I_n transformations form a group.

	T_m	I_m
T_n	T_{m+n}	I_{m+n}
I_n	I_{n-m}	T_{n-m}

Table 2.10. The binary composition of the T_n/I_n group.

Because the neo-Riemannian (*G*) and T_n/I_n (*T*) groups are isomorphic to one another, we also know that there will be a normal subgroup (*N*) of the T_n/I_n group whose left and right cosets will be the same and that will generate the quotient group *T*/*N*. This normal subgroup is the set $\{T_0, T_4, T_8\}$, and all eight of its cosets are displayed in Table 2.11.

$\{T_0, T_4, T_8\}$

$\{T_1, T_5, T_9\}$

$\{T_2, T_6, T_{10}\}$

$\{T_3, T_7, T_{11}\}$

$\{I_0, I_4, I_8\}$

$\{I_1, I_5, I_9\}$

$\{I_2, I_6, I_{10}\}$

$\{I_3, I_7, I_{11}\}$

Table 2.11. The cosets of $\{T_0, T_4, T_8\}$ in the T_n/I_n group.

The big question, then, is whether or not this quotient group is also isomorphic to the Z*n*/W*n* group as the neo-Riemannian quotient group was to the Y*n*/X*n* group. To see if this is so, let us consider the effect of the cosets at the level of the SUM classes: D+ inhabits $\boxed{5}$, T_3(D+) = F+, T_7(D+) = A+, T_{11}(D+) = C#+, and F+, A+, and C# all inhabit $\boxed{2}$. The Z*n*/W*n* transformation that maps $\boxed{5}$ to $\boxed{2}$ is Z9. In other words, the triadic mappings achieved by the coset $\{T_3, T_7, T_{11}\}$ produces mappings at the level of the SUM classes that are identical to the actions of Z9. Similarly, the coset $\{I_0, I_4, I_8\}$ produces triadic mappings that result in the same SUM-class mappings as W0. When we compose $\{T_3, T_7, T_{11}\}$ with $\{I_0, I_4, I_8\}$ we get $\{I_3, I_7, I_{11}\}$, whose actions correspond to W9 at the level of the SUM classes, and Z9 ∘ W0 is also W9. We can thus define the homomorphism from the T_n/I_n group onto the Z*n*/W*n* group as seen in Table 2.12.

$\mathbf{T_n/I_n}$ **Transformations**	**Isomorphism**	***Zn/Wn*** **Transformations**
$\{T_0, T_4, T_8\}$	⇔	Z0
$\{T_1, T_5, T_9\}$	⇔	Z3
$\{T_2, T_6, T_{10}\}$	⇔	Z6
$\{T_3, T_7, T_{11}\}$	⇔	Z9
$\{I_0, I_4, I_8\}$	⇔	W0
$\{I_1, I_5, I_9\}$	⇔	W3
$\{I_2, I_6, I_{10}\}$	⇔	W6
$\{I_3, I_7, I_{11}\}$	⇔	W9

Table 2.12. The homomorphism from the T_n/I_n group onto the Z*n*/W*n* group mediated by the isomorphism between the quotient group of the T_n/I_n group modulo $\{T_0, T_4, T_8\}$ and the Z*n*/W*n* group.

The T_n/I_n group can also be used to make a GIS analogous to the neo-Riemannian GIS we defined earlier. Here, once again, *S* is the set of all major and minor triads, *G* the T_n/I_n group, and int a mapping from pairs of elements in *S* to *G*. This marks the fourth GIS we have created so far, and one might well begin to wonder at the analytical value of so many different systems. Recall that we first began this long exploration of commuting groups because the Y*n*/X*n* and neo-Riemannian transformations obscured voice-leading equivalence when it occurred between triads of different quality. The new Z*n*/W*n* group offers some distinct advantages because it allows us to make use of the T_n transformations at the level of the triad, which we noted are consistent regardless of the quality of the triad they are applied to. But this system also has its disadvantages. Indeed, referring again to the mapping table for the Z*n*/W*n* transformations (Table 2.8), we can note that while the W*n* transformations do act simply transitively on the SUM classes, they do so in a way that does not produce consistent intervals for each transformation. Consider W0, for example, which maps [1] to [11] (and vice versa) but also [2] to [10], [4] to [8], and [5] to [7]. Each of these pairs lies the same interval from a particular axis within the SUM-class universe (0, which is not part of the space for the consonant triads), but this also means that the pairs do not lie the same interval from each other.

The implications of this at the level of the triads is that triads related by I_n transformations that belong to the same SUM-class transformation will *not* lie the same PVLS interval from one another. For example, transforming D+ by I_0 returns Eb-, and PVLS(D+, Eb-) = 2. But when I_0 is applied to G+, it returns Bb-, which lies a PVLS interval of 6 from G+. In other words, what is "congruent" about the I_n transformations within each equivalence class is not the voice-leading intervals they produce between triads, but rather the voice-leading intervals they produce between a triad and a given inversional axis. For a model aimed at generalizing voice-leading intervals between all consonant triads, then, neither the W*n* or I_n transformations are particularly useful. If this is the case, then why did we bother to invoke this other set of GISs at all?

Recall that each of the four groups of transformations we have created so far have consisted of two different "types" of transformations—transpositions and inversions. From the Y*n*/X*n* and neo-Riemannian groups, we found that the inversions (the X*n* transformations and all odd-numbered parings of P, L, or R) meshed well with our usual notions of how an inversion should operate and also were able to generalize voice-leading intervals, whereas the transpositions were not particularly intuitive or meaningful. Conversely, it is the transpositions of the Z*n*/W*n* and T_n/I_n groups that are most meaningful for our model. In other words, we have a perfect set of transpositions and a perfect set of inversions at both the level of the triad and the SUM class between these various GISs. But just because these transformations inhabit different groups does not necessarily mean that they cannot be mixed and matched with one another.

Consider for example, the relationships between all of these GISs as seen in Figure 2.1. What this figure reveals, above all, is that the common ground between isomorphic (to abuse the language slightly) GISs is exactly the spaces they act upon. We noted earlier that all of these groups acted simply transitively upon their respected spaces, which means that there is only ever one way to get from any element in the space to each of the other elements in the space. To be more specific, there is never any overlap in the action of two transformations. This means that all of the transpositionally-related elements in a space would be left totally untouched if we took away all of the transformations that acted upon that space (and likewise for the inversions), and furthermore that we could then substitute in another set of transpositions without "stepping on the toes" of the inversions. In other words, if we took all of the transpositions from one simply-transitive group and all of the inversions from another simply-transitive group that acted upon the same space, this new set (NB, not necessarily a group) of transformations would also act simply-transitively upon that same space! Therefore, we can take the T_n transformations from the T_n/I_n group, the neo-Riemannian inversions from the neo-Riemannian group, the Z*n* transformations from the Z*n*/W*n* group, and the X*n* transformations from the Y*n*/X*n* group and create a new set of triadic transformations and a new set of SUM-class transformations. As it happens, neither of these sets is closed under any binary composition,[44] and so we will have to be careful to remember that this is only a *set* of transformations and not a proper group. But through the combined resources of all the GISs in Figure 2.1, we can create a transformational system that intuitively models all voice-leading intervals between triads and their equivalence classes.

[44] This because each of the original groups are actually part of an enormous group that Lewin calls PETINV, which contains " . . . all operations on S that can be expressed as functionally equal to something of the form PT, where P is some interval-preserving operation and T is some transposition. . . . plus the family INVS of all inversion operations" (Lewin, GMIT, 57). This full group is not only extremely large, but many of these transformations also behave in rather strange ways that are not particularly useful for our present contexts.

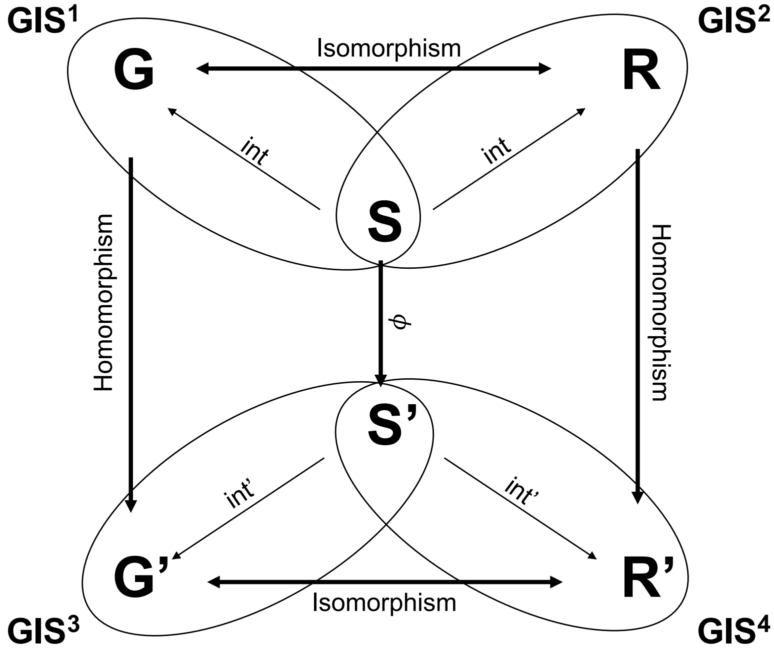
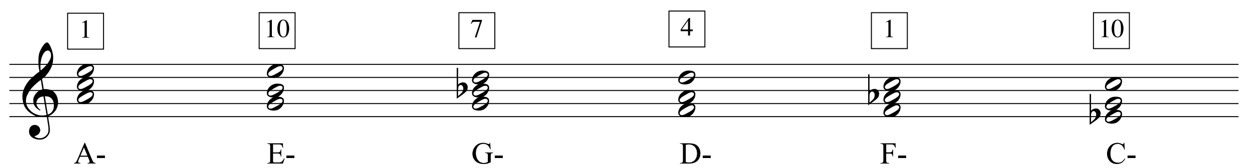


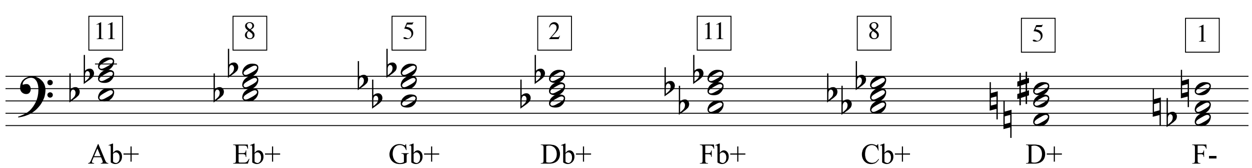
Figure 2.1. A visual representation of four related GISs where G is the group of T_n/I_n transformations, R the group of neo-Riemannian transformations, S the major and minor triads, G' the group of Z*n*/W*n* transformations, R' the group of Y*n*/X*n* transformations, S' the SUM classes, and ϕ a three-to-one mapping from S to S'. Each instance of int and int' maps S × S and S' × S' respectively to various transformational groups in the manner of Definition 2.2.

To see this system in action, let us now return to Stanford's *La belle dame*. This dramatic ballad on a poem by John Keats tells the story of an ill-fated knight who meets a woman with whom he seems to fall instantly in love. After a day full of joyous adventure with her, he eventually finds himself at her home where he is lulled to sleep. While he sleeps, the knight dreams a strange and terrible dream in which he sees ghosts of other unlucky men that have fallen victim to the wiles of this "*belle dame sans merci*." The rather commonplace harmonic language that dominates the majority of the ballad suddenly gives way to an extremely chromatic sequence of harmonies at measure 98 as the knight begins the account of his dream. A very similar sequence begins again in measure 122 as the knight recounts his sudden and rude awakening from the dream to find himself on a cold, empty hillside. Fétis speaks of sequences as a sort of temporary "suspension" of a tonal reality, which seems to mesh well with Stanford's use of the sequences here to portray the suspension of reality that occurs during a dream.[45]

We examined parts of these sequences earlier (reproduced below as Examples 2.4 and 2.5) as an example of some of the shortcomings inherent within the neo-Riemannian and Y*n*/X*n* groups. But now when examined through the lens of the T_n and *Zn* transformations as seen in Figures 2.2 and 2.3, the equivalence of these two sequences is captured quite nicely. Furthermore, through the power of the dual transformational groups, we may also use the X*n* transformations to relate these two progressions directly to one another as seen in Figure 2.4. Thus, although these two progressions do not occur together in time, their voice-leading equivalence and their role as a frame for the knight's dream suggests a connection between them.



Example 2.4. A harmonic reduction of Stanford, *La belle dame sans merci*, mm. 98–108.



Example 2.5. A harmonic reduction of Stanford, *La belle dame sans merci*, mm. 122–130.

[45] See François-Joseph Fétis, *Esquisse de l'histoire de l'harmonie: An English-Language Translation of the François-Joseph Fétis History of Harmony*, trans. Mary I. Arlin (Stuyvesant: Pendragon Press, 1994), 164.

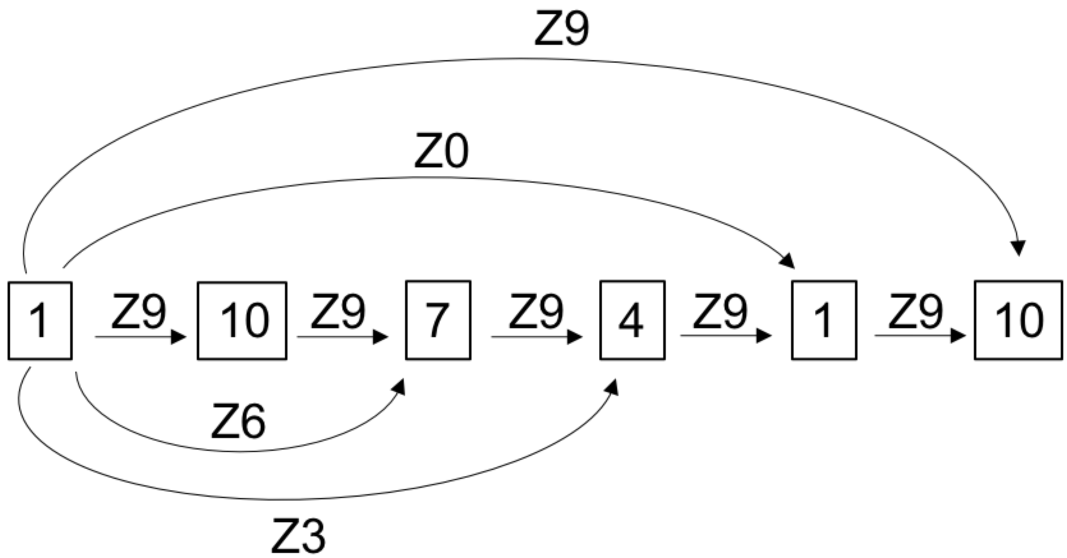


Figure 2.2. A transformational network of Stanford, *La belle dame sans merci*, mm. 98–108.

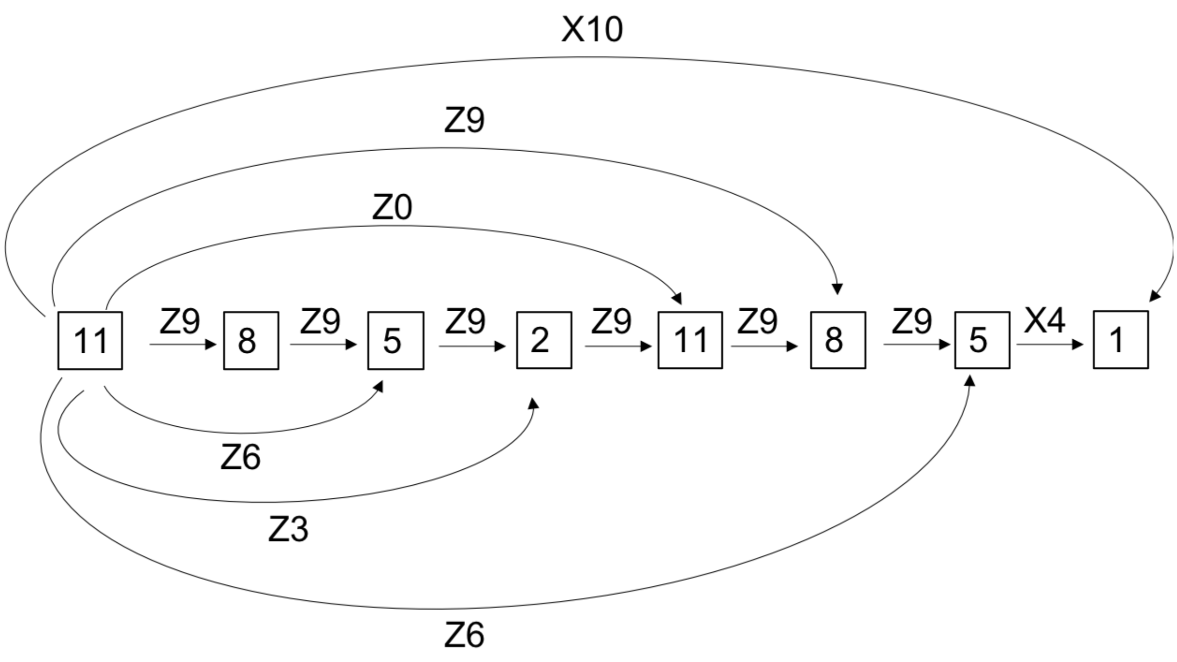


Figure 2.3. A transformational network of Stanford, *La belle dame sans merci*, mm. 122–130.

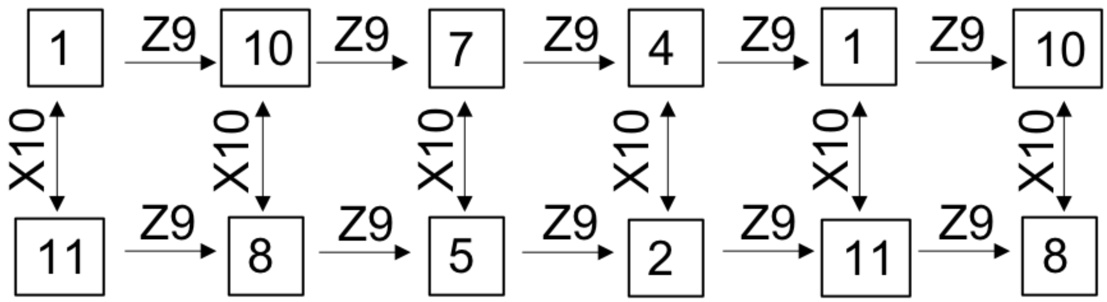


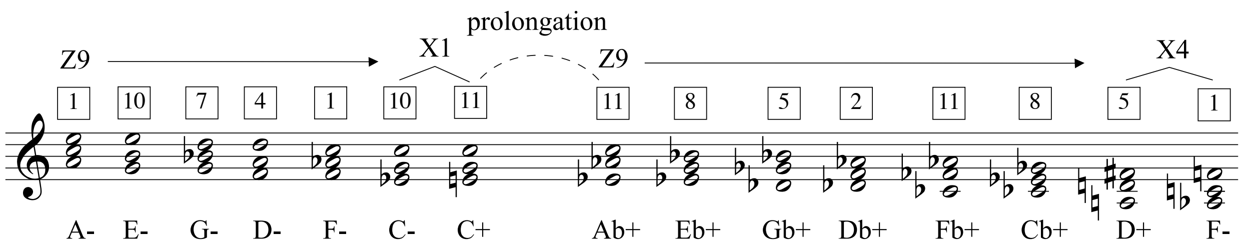
Figure 2.4. A commutative transformational network relating mm. 98–108 and 122–128 of Stanford, *La belle dame sans merci*.

In addition to relating the two sequences to one another, X10 also connects the beginning of each sequence with the triads of opposite quality that break the two sequences (C+ in 109 and F- in 130). Furthermore, the C-major triad that ends the first sequence in measure 109 belongs to the same SUM class (11) as the Ab-major triad that begins the second sequence in measure 122. The measures between these two chords, which are much less consistent from a voice-leading perspective, might thus be thought of as a prolongation of 11.

That the end of the first sequence and the beginning of the second belong to the same SUM class also suggests that these two sequences might be connected together into one large chain. Referring again to Examples 2.4 and 2.5, we can see that both sequences produce a consistent descent by whole step in the soprano and tenor voices and by half step in the alto.[46] The implied voicing shown in Example 2.6 nicely demonstrates the connection between the two sequences: after the interruption of the C major triad in measure 99 and the prolongation of 11, the soprano and tenor pick up where they left off (an octave higher) in measure 122 and continue

[46] It should be noted, once again, that the voicing of the C- minor triad at the end of Example 2.4 represents the implied continuation of the voice-leading pattern. The chord actually contains Eb in the highest voice in measure 108.

on their whole-step descent. Had the sequence been allowed to continue for one more iteration, it would have completed the whole-tone scales in the soprano and tenor (the two unique whole-tone scales), the chromatic scale in the alto, and brought us back to an A-rooted triad. In other words, the whole sequence could have started all over again. Instead, however, we overshoot by a single semitone (descending by a major third instead of a minor third), which wrenches us free from the sequence (and thus the dream) at the last possible moment and returns us to the "reality" of the ballad's home key of F minor.



Example 2.6. A reduction and analysis of Stanford, *La belle dame sans merci*, mm. 98–130.

The implications of this for the narrative of the ballad are profound. This sequence, and the never-ending cycle of *la belle dame's* deception that it represents could have continued forever. This knight might have been just another ghost of a poor unfortunate man deceived by this woman. This time, however, the ghosts of the men she has deceived in the past are able to break the cycle just like the C-major triad first breaks the sequence in measure 109 as the ghosts cry out to warn the knight of his fate, which helps the knight to break free. In this way, voice leading actually plays a role in articulating the narrative of the ballad—a role that we would not have been able to observe as readily without the system we constructed in this chapter.

Chapter 3: SUM Classes for Other Set Classes

Chapter 2 explored the SUM-class system within the context of the major and minor triads (set class 3-11). This is the context for which this system was originally designed by Richard Cohn, but recent work by Joseph Straus reveals that at least the SUM classes themselves are easily extensible to other set classes as well.[47] While perhaps not as immediately useful in analytical contexts as the system of classes on the consonant triads (since there is less repertoire that makes use of a single set class in the way that tonal music makes use of the consonant triads), these systems do reveal interesting properties of the set classes themselves and the SUM-class system generally. We begin by exploring how SUM classes behave on the remainder of the trichordal set classes before moving on to examine these structures on larger-cardinality set classes.

3.1 SUM Classes for 3-2, 3-3, 3-5, 3-7, and 3-8

As with the consonant triads, constructing a SUM-class system for any other set class is simply a matter of finding the SUM values of all the members of that set class and placing those sets that share the same values into the same SUM class. In fact, the structure of the SUM-class spaces for set classes 3-2, 3-3, 3-5, 3-7, and 3-8 are identical to the space we observed for in the consonant triads. These spaces are displayed in Tables 3.1 through 3.5.

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[47] See Cohn, "Square Dances" and Joseph N. Straus, "Sum Class," *Journal of Music Theory* 62, no. 2 (2018): 279–338.

SUM Class	Pitch-Class Set Members
1	{{3, 4, 6}, {7, 8, 10}, {11, 0, 2}}
2	{{11, 1, 2}, {3, 5, 6}, {7, 9, 10}}
4	{{0, 1, 3}, {4, 5, 7}, {8, 9, 11}}
5	{{0, 2, 3}, {4, 6, 7}, {8, 10, 11}}
7	{{1, 2, 4}, {5, 6, 8}, {9, 10, 0}}
8	{{9, 11, 0}, {1, 3, 4}, {5, 7, 8}}
10	{{2, 3, 5}, {6, 7, 9}, {10, 11, 1}}
11	{{10, 0, 1}, {2, 4, 5}, {6, 8, 9}}

Table 3.1. The SUM classes of set class 3-2.

SUM Class	Pitch-Class Set Members
1	{{10, 1, 2}, {2, 5, 6}, {6, 9, 10}}
2	{{3, 4, 7}, {7, 8, 11}, {11, 0, 3}}
4	{{11, 2, 3}, {3, 6, 7}, {7, 10, 11}}
5	{{0, 1, 4}, {4, 5, 8}, {8, 9, 0}}
7	{{8, 11, 0}, {0, 3, 4}, {4, 7, 8}}
8	{{1, 2, 5}, {5, 6, 9}, {9, 10, 1}}
10	{{9, 0, 1}, {1, 4, 5}, {5, 8, 9}}
11	{{2, 3, 6}, {6, 7, 10}, {10, 11, 2}}

Table 3.2. The SUM classes of set class 3-3.

SUM Class	Pitch-Class Set Members
1	{{2, 3, 8}, {6, 7, 0}, {10, 11, 4}}
2	{{9, 2, 3}, {1, 6, 7}, {5, 10, 11}}
4	{{3, 4, 9}, {7, 8, 1}, {11, 0, 5}}
5	{{6, 11, 0}, {10, 3, 4}, {2, 7, 8}}
7	{{0, 1, 6}, {4, 5, 10}, {8, 9, 2}}
8	{{7, 0, 1}, {11, 4, 5}, {3, 8, 9}}
10	{{1, 2, 7}, {5, 6, 11}, {9, 10, 3}}
11	{{8, 1, 2}, {0, 5, 6}, {4, 9, 10}}

Table 3.3. The SUM classes of set class 3-5.

SUM Class	Pitch-Class Set Members
1	{{2, 4, 7}, {6, 8, 11}, {10, 0, 3}}
2	{{10, 1, 3}, {2, 5, 7}, {6, 9, 11}}
4	{{3, 5, 8}, {7, 9, 0}, {11, 1, 4}}
5	{{7, 10, 0}, {11, 2, 4}, {3, 6, 8}}
7	{{0, 2, 5}, {4, 6, 9}, {8, 10, 1}}
8	{{8, 11, 1}, {0, 3, 5}, {4, 7, 9}}
10	{{1, 3, 6}, {5, 7, 10}, {9, 11, 2}}
11	{{9, 0, 2}, {1, 4, 6}, {5, 8, 10}}

Table 3.4. The SUM classes of set class 3-7.

SUM Class	**Pitch-Class Set Members**
1	{{9, 1, 3}, {1, 5, 7}, {5, 9, 11}}
2	{{2, 4, 8}, {6, 8, 0}, {10, 0, 4}}
4	{{6, 10, 0}, {10, 2, 4}, {2, 6, 8}}
5	{{3, 5, 9}, {7, 9, 1}, {11, 1, 5}}
7	{{7, 11, 1}, {11, 3, 5}, {3, 7, 9}}
8	{{0, 2, 6}, {4, 6, 10}, {8, 10, 2}}
10	{{8, 0, 2}, {0, 4, 6}, {4, 8, 10}}
11	{{1, 3, 7}, {5, 7, 11}, {9, 11, 3}}

Table 3.5. The SUM classes of set class 3-8.

As with the consonant triads, each of these set classes contains twenty-four unique sets; twelve sets analogous to the twelve minor triads (which we shall call "prime" sets since they include the prime-form representative of the set class and all of its transpositions) and twelve sets analogous to the twelve major triads (which we shall call "inverted" sets since they include all inversions of the set class's prime form). These two different forms of the set class are segregated into separate SUM classes just as we saw with the minor and major triads in the SUM-class system for the consonant triads. Also like the consonant triads, each SUM class contains the three pitch-class sets whose members belong to the same augmented triad as the members of the other sets at the same order position. For example, in the SUM-class system for 3-3 (Table 3.2), the "first" pitch classes of {10, 1, 2}, {2, 5, 6}, and {6, 9, 10} (all members of 1) all belong to the same augmented triad, and likewise for pitch classes at order positions two and three. In some cases (3-3 and 3-8) the three sets in the same SUM class share common tones like the major and minor triads did. The most significant parallel to the consonant triad system, however, is the fact that these SUM classes also generalize PVLS intervals! For example, {10, 1, 2}, {2, 5, 6}, and {6, 9, 10} from 1 of the 3-3 system are all separated by a PVLS interval of 0: PVLS({10, 1, 2}, {2, 5, 6}) = (2 – 10) + (5 – 1) + (6 – 2) = 4 + 4 + 4 = 0. Indeed, this should not surprise us because Cohn's proof from the introduction showed that a PVLS is directly related to SUM values. Moreover, this proof was not dependent in any way upon the set-class membership of the pitch-class sets involved, and so we can know that any set class that generates the same SUM-class system as the consonant triads will also possess the same kinds of voice-leading relationships between its elements.

This also means, of course, that we can use the same SUM-class transformation groups on any SUM-class systems that are the same. Thus, the Y*n*/X*n* and Z*n*/W*n* groups from Chapter 2 will also be operative on the SUM-class spaces of 3-2, 3-3, 3-5, 3-7, and 3-8 (these groups and their actions upon the SUM classes are reproduced below as Tables 3.6 and 3.7 for reference). Additionally, the issues with the contextuality of the Y*n* transformations and the inconsistent intervals by the W*n* transformations will also apply here as well. If we wish to create a system for these set classes like the one we used for the consonant triads, then, we will want to be able to use the X*n* transformations to generalize voice leading between inversionally-related sets and the Z*n* transformations for voice leading between transpositionally-related sets. We can also know that the Y*n*/X*n* and Z*n*/W*n* groups will always be isomorphic to the quotient groups of the neo-Riemannian and T_n/I_n groups regardless of the sets these transformations they act upon because of the group-theory work we did in Chapter 2. The question here, then, is whether or not these quotient groups produce meaningful actions on any of these other set classes.

Y*n*/X*n* SUM-Class Transformation	**Action on SUM Classes**
Y0	(1) (2) (4) (5) (7) (8) (10) (11)
Y3	(1, 4, 7, 10) (2, 11, 8, 5)
Y6	(1, 7) (2, 8) (4, 10) (5, 11)
Y9	(1, 10, 7, 4) (2, 5, 8, 11)
X1	(1, 2) (4, 5) (7, 8) (10, 11)
X4	(1, 5) (2, 10) (4, 8) (7, 11)
X7	(1, 8) (2, 7) (4, 11) (5, 10)
X10	(1, 11) (2, 4) (5, 7) (8, 10)

Table 3.6. The permutations of the SUM classes achieved by the Y*n*/X*n* SUM-class transformations.

***Zn*/W*n* SUM-class Transformations**	**Action on Sum Classes**
Z0	(1) (2) (4) (5) (7) (8) (10) (11)
Z3	(1, 4, 7, 10) (2, 5, 8, 11)
Z6	(1, 7) (2, 8) (4, 10) (5, 11)
Z9	(1, 10, 7, 4) (2, 11, 8, 5)
W0	(1, 11) (2, 10) (4, 8) (5, 7)
W3	(1, 2) (4, 11) (5, 10) (7, 8)
W6	(1, 5) (2, 4) (7, 11) (8, 10)
W9	(1, 8) (2, 7) (4, 5) (10, 11)

Table 3.7. The permutations of the SUM classes achieved by the Z*n*/W*n* SUM-class transformations.

In order to test this, we must simply see whether all three transformations in a few randomly-selected cosets will actually map between sets that belong to the same SUM classes. If we find this to be true for one SUM class, then the congruence of the SUM classes will guarantee that this will also be true for all other SUM classes of the same set class. Table 3.8 does just this, and, as can be seen, the transformations within the same cosets do indeed produce consistent voice-leading intervals within each set class. Thus, the T_n/I_n quotient group generated from the normal subgroup $\{T_0, T_4, T_8\}$ is a meaningful voice-leading generalization not just for the consonant triads, but also for (at least) set classes 3-2, 3-3, 3-5, 3-7, and 3-8. This quotient group and the homomorphism mapping it onto the Z*n*/W*n* group is reproduced in Table 3.9.

Set Class	Cosets	SUM Class
3-2	$T_1(\{3, 4, 6\}) = \{4, 5, 7\}$ $T_5(\{3, 4, 6\}) = \{8, 9, 11\}$ $T_9(\{3, 4, 6\}) = \{0, 1, 3\}$	{{0, 1, 3}, {4, 5, 7}, {8, 9, 11}} = $\boxed{4}$
3-3	$T_2(\{10, 1, 2\}) = \{0, 3, 4\}$ $T_6(\{10, 1, 2\}) = \{4, 7, 8\}$ $T_{10}(\{10, 1, 2\}) = \{8, 11, 0\}$	{{8, 11, 0}, {0, 3, 4}, {4, 7, 8}} = $\boxed{7}$
3-5	$I_0(\{2, 3, 8\}) = \{10, 9, 4\}$ $I_4(\{2, 3, 8\}) = \{2, 1, 8\}$ $I_8(\{2, 3, 8\}) = \{6, 5, 0\}$	{{8, 1, 2}, {0, 5, 6}, {4, 9, 10}} = $\boxed{11}$
3-7	$I_3(\{2, 4, 7\}) = \{1, 11, 8\}$ $I_7(\{2, 4, 7\}) = \{5, 3, 0\}$ $I_{11}(\{2, 4, 7\}) = \{9, 7, 4\}$	{{8, 11, 1}, {0, 3, 5}, {4, 7, 9}} = $\boxed{8}$
3-8	$T_3(\{9, 1, 3\}) = \{0, 4, 6\}$ $T_7(\{9, 1, 3\}) = \{4, 8, 10\}$ $T_{11}(\{9, 1, 3\}) = \{8, 0, 2\}$	{{8, 0, 2}, {0, 4, 6}, {4, 8, 10}} = $\boxed{10}$

Table 3.8. The mappings achieved when the transformations of the same $\{T_0, T_4, T_8\}$ coset are applied to the same pitch-class set.

T_n/I_n Transformations	Isomorphism	*Zn*/W*n* Transformations
$\{T_0, T_4, T_8\}$	⇔	Z0
$\{T_1, T_5, T_9\}$	⇔	Z3
$\{T_2, T_6, T_{10}\}$	⇔	Z6
$\{T_3, T_7, T_{11}\}$	⇔	Z9
$\{I_0, I_4, I_8\}$	⇔	W0
$\{I_1, I_5, I_9\}$	⇔	W3
$\{I_2, I_6, I_{10}\}$	⇔	W6
$\{I_3, I_7, I_{11}\}$	⇔	W9

Table 3.9. The homomorphism from the T_n/I_n group onto the Z*n*/W*n* group mediated by the isomorphism between the quotient group of the T_n/I_n group modulo $\{T_0, T_4, T_8\}$ and the Z*n*/W*n* group.

To make use of the neo-Riemannian group for pitch-class sets other than the major and minor triads will require that we reconceptualize these transformations slightly. Instead of defining their actions in references to things like relatives, parallels, or mediants (which have no meaning outside of the world of tonal harmony), let us define these transformations as inversions that hold particular portions of the set invariant. Consider, for example, the action of the P transformation on D+: P({2, 6, 9}) = {2, 5, 9}. In terms of a triad, we would say that this transformation lowers the third of the chord by semitone, but in terms of a pitch-class set we would say that there is some transformation (I_n, since these triads are of opposite quality) that holds 2 and 9 invariant while sending 6 to 5. One surefire way to guarantee that two pitch classes are preserved is to map them to each other. Recall that the I_n transformations act according to the rule $I_n(x) = n - x$, and so any n derived from the sum (mod 12) of two pitch classes will always map those two pitch classes to one another. Thus, I_{11} will send 2 to 9, 9 to 2, *as well as* 6 to 5.

But this does not mean that I_{11} is synonymous with P. I_{11} of C+, for example, is not C- but E-, and I_{11} of G+ is A-. To consistently retrieve a I_n transformation that will return a parallel triad, then, we need to define it according to the axis *within the set* that we want to invert about. For P, we know we always want to preserve the “root” (the pitch class at order-position one within the set) and “fifth” (the pitch class at order-position three within the set), and so we can simply invert about the axis derived from their sum. In other words, P~ (generalized) is the transformation I_n such that n is the sum of the first and last pitch classes in the set. Thus, P~ of A+ ({9, 1, 4}) = $I_{9+4} = I_1$; and I_1({9, 1, 4}) = {4, 0, 9} (A-). When defined this way, these transformations can act on any three-note set. For example, P~ of {3, 4, 6} (a set from 3-2) is $I_{3+6} = I_9$, and I_9({3, 4, 6}) = {6, 5, 3}—also a set from 3-2.

The work of Morris and Straus shows that we can do likewise for L and R as well,[48] but in these cases we will have to be careful to specify whether the transformation is acting upon a prime or inverted form of the set. We do this for the usual L and R as well but probably without noticing because it is natural to think that if X is related to Y, that Y will be related to X. In terms of a transformation or mapping, though, any transformation that takes X to Y and also Y to X will have to move in opposite directions depending on context. In the case of triads, we know that a major triad's relative is its submediant, but that a minor triad's relative is its mediant.

Straus's formal definitions of the generalized P~, L~, and R~ can be seen in Definitions 3.1 through 3.3.[49] Note well the notation of the sets with angle brackets (< >), which indicates a particular ordering.

Definition 3.1. $P\sim(\langle a, b, c\rangle) = I_{a+c}$.

Definition 3.2. $L\sim(\langle a, b, c\rangle) = I_{a+b}$ if the set is prime or I_{b+c} if the set is inverted.

Definition 3.3. $R\sim(\langle a, b, c\rangle) = I_{b+c}$ if the set is prime or I_{a+b} if the set is inverted.

It is essential that we be specific about the ordering of these sets because these transformations derive their inversional axes from the sum of pitch classes at specific order positions. P~ of a set in two different orderings would thus produce two very different results. For example, P~ of {2, 6, 9} = $I_{2+9} = I_{11}$ whereas P~ of {6, 2, 9} = $I_{6+9} = I_3$. For the purposes of this paper, the

[48] See Robert D. Morris, "Voice-Leading Spaces," *Music Theory Spectrum* 20, no. 2 (1998): 175–208; Joseph N. Straus, "Contextual-Inversion Spaces," *Journal of Music Theory* 55, no. 1 (2011): 43–88. These transformations are only intended to be used on non-symmetrical sets, and therefore will not necessarily produce meaningful transformations on symmetrical sets.

[49] Adapted from Straus, "Contextual-Inversion Spaces," 54. Straus also defines the actions of R', P', and L', which invert sets about their first, second, and third order positions respectively. The same results can be obtained by the composition of the P, L, and R transformations, however, and so I have elected to use only P, L, and R and for the sake of simplicity.

contextual inversions are defined to act on prime forms of a set class in Rahn's normal order[50] and on inverted forms of the set class in the *order that results when an inverted set is derived from the retrograde inversion of a normal-order prime*.[51] This is a mouthful, but what it means is that if we were to invert a prime-form set in normal order, the resulting set would already in the correct order for the contextual inversion but read backwards. This makes the process of inversion much simpler when these contextual inversions are chained together (which we shall often have cause to do). Otherwise, we would have to stop and calculate the normal order of each set before continuing on. In many cases, the normal order of an inverted set and the retrograde inversion of a prime set in normal order are the same, but for the fifty sets seen in Table 3.10 these two orderings are actually different. If the contextual inversions were defined to act on inverted sets in normal order, then, they would behave differently on these fifty sets.

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[50] The normal-order algorithms used by John Rahn (*Basic Atonal Theory* (New York: Schirmer, 1980)) and Allen Forte (*The Structure of Atonal Music* (New Haven: Yale University Press, 1973)) are slightly different when two orderings of a set produce the same outer interval. Whereas Rahn's algorithm chooses sets that are most left-packed from the right, Forte chooses sets most left packed from the left. Despite these differences, however, the methods only produce different normal orders for set classes 5-20, 6-z29, 6-31, and 7-20.

[51] In the fourth edition of *Introduction to Post-Tonal Theory*, Straus modifies the normal-order algorithm slightly so that the ordering that is most packed to the left *or* the right is chosen. Straus indicates that this changes the normal form for a small number of sets (including 4-19) but does not specify exactly which ones. See Joseph N. Straus, *Introduction to Post-Tonal Theory*, *4th ed.*, (New York: W. W. Norton, 2016), 45–46.

Three	Four	Five	Six	Seven	Eight	Nine
3-9	4-6	5-13	6-27	7-10	8-3	9-6
	4-19	5-15	6-z28	7-z12	8-7	9-7
	4-20	5-z17	6-z37	7-16	8-8	9-8
	4-24	5-31	6-z38	7-z17	8-21	9-9
		5-32	6-z42	7-19	8-22	9-10
		5-34	6-z44	7-21	8-23	9-11
		5-35	6-z45	7-22	8-24	
			6-z46	7-33	8-25	
			6-z47	7-34	8-26	
			6-z48	7-35	8-27	
			6-z49			
			6-z50			

Table 3.10. The 50 set classes for which the inversion of a prime set in normal order does not produce the retrograded normal order of the inverted set.

Because these generalized neo-Riemannian transformations are defined contextually, they will behave slightly differently for each set class they are applied to. Within the universe of the major and minor triads, for example, P transforms the triads by moving the third of the chord up or down by semitone—a maximally-smooth transformation in Cohn's nomenclature.[52] Within the 3-3 set class, however, there are no two sets with the same outer pitch classes ("root" and

[52] See Richard Cohn, "Maximally Smooth Cycles, Hexatonic Systems, and the Analysis of Late-Romantic Triadic Progressions," *Music Analysis* 15, no. 1 (1996): 9–40.

“fifth”) whose middle pitch classes are only a semitone apart. P~, therefore, always transforms the middle pitch class by whole tone rather than semitone. This causes these generalized transformations to behave in ways that are very different from the usual neo-Riemannian transformations and also to compose in strange and unusual ways. A group of P~, L~, and R~ transformations would thus look different for every set class even though the actions of the whole group on the space would be the same.

Toward the end of a unified group of contextual transformations for all pitch-class sets, let us define two variable transformations Δ^n and Ω where Δ^n is some combination of P~, L~, and/or R~ that transposes an entire prime-form set up by semitone (and an inverted-form set down by semitone) and Ω some combination of P~, L~, and R~ that acts as an inversion and takes any set to a set in the nearest SUM class “above.” The twelve powers of Δ^n and $\Delta^n\Omega$ (where $\Delta^{12}\Omega = \Omega$) then form a group where Δ^{12} is the group identity, the inverse of any Δ^n is Δ^{12-n}, and $\Delta^n\Omega$ is its own inverse. If Δ^1 is understood to be equivalent to T_1 (when applied to a prime form) and Ω a generic inversion, then it is easy to see that this group is isomorphic to the T_n/I_n group and thus is also a group.[53] The set $\{\Delta^{12}, \Delta^4, \Delta^8\}$ is a normal subgroup of this group, and from it we may generate eight cosets seen in Table 3.11.

[53] In fact, I_n is more properly understood to be the compound transformation T_nI where I is a generic inversion (usually about 0). Written this way, the relationship between these two groups is even clearer.

$\{\Delta^{12}, \Delta^4, \Delta^8\}$

$\{\Delta^1, \Delta^5, \Delta^9\}$

$\{\Delta^2, \Delta^6, \Delta^{10}\}$

$\{\Delta^3, \Delta^7, \Delta^{11}\}$

$\{\Delta^{12}\Omega\ (\Omega),\ \Delta^4\Omega, \Delta^8\Omega\}$

$\{\Delta^1\Omega,\ \Delta^5\Omega, \Delta^9\Omega\}$

$\{\Delta^2\Omega,\ \Delta^6\Omega, \Delta^{10}\Omega\}$

$\{\Delta^3\Omega,\ \Delta^7\Omega, \Delta^{11}\Omega\}$

Table 3.11. The eight cosets of $\{\Delta^{12}, \Delta^4, \Delta^8\}$ in the $\Delta^n/\Delta^n\Omega$ group.

These transformations and cosets are meaningless, however, until we assign them a particular value for each set class, and as such we cannot yet see if this quotient group is actually isomorphic to the Y*n*/X*n* like we would want it to be. To investigate this, let us begin with set class 3-2, and assign Δ^1 the value P~R~ and Ω the value P~. Since this is the first time we have made use of these transformations, let us walk through the process step by step by applying these transformations to the prime-form representative of the 3-2 set class, {0, 1, 3}, which inhabits 4 (the SUM-class space for which is reproduced below as Table 3.12).

SUM Class	Pitch-Class Set Members
[1]	{{3, 4, 6}, {7, 8, 10}, {11, 0, 2}}
[2]	{{11, 1, 2}, {3, 5, 6}, {7, 9, 10}}
[4]	{{0, 1, 3}, {4, 5, 7}, {8, 9, 11}}
[5]	{{0, 2, 3}, {4, 6, 7}, {8, 10, 11}}
[7]	{{1, 2, 4}, {5, 6, 8}, {9, 10, 0}}
[8]	{{9, 11, 0}, {1, 3, 4}, {5, 7, 8}}
[10]	{{2, 3, 5}, {6, 7, 9}, {10, 11, 1}}
[11]	{{10, 0, 1}, {2, 4, 5}, {6, 8, 9}}

Table 3.12. The SUM classes of set class 3-2.

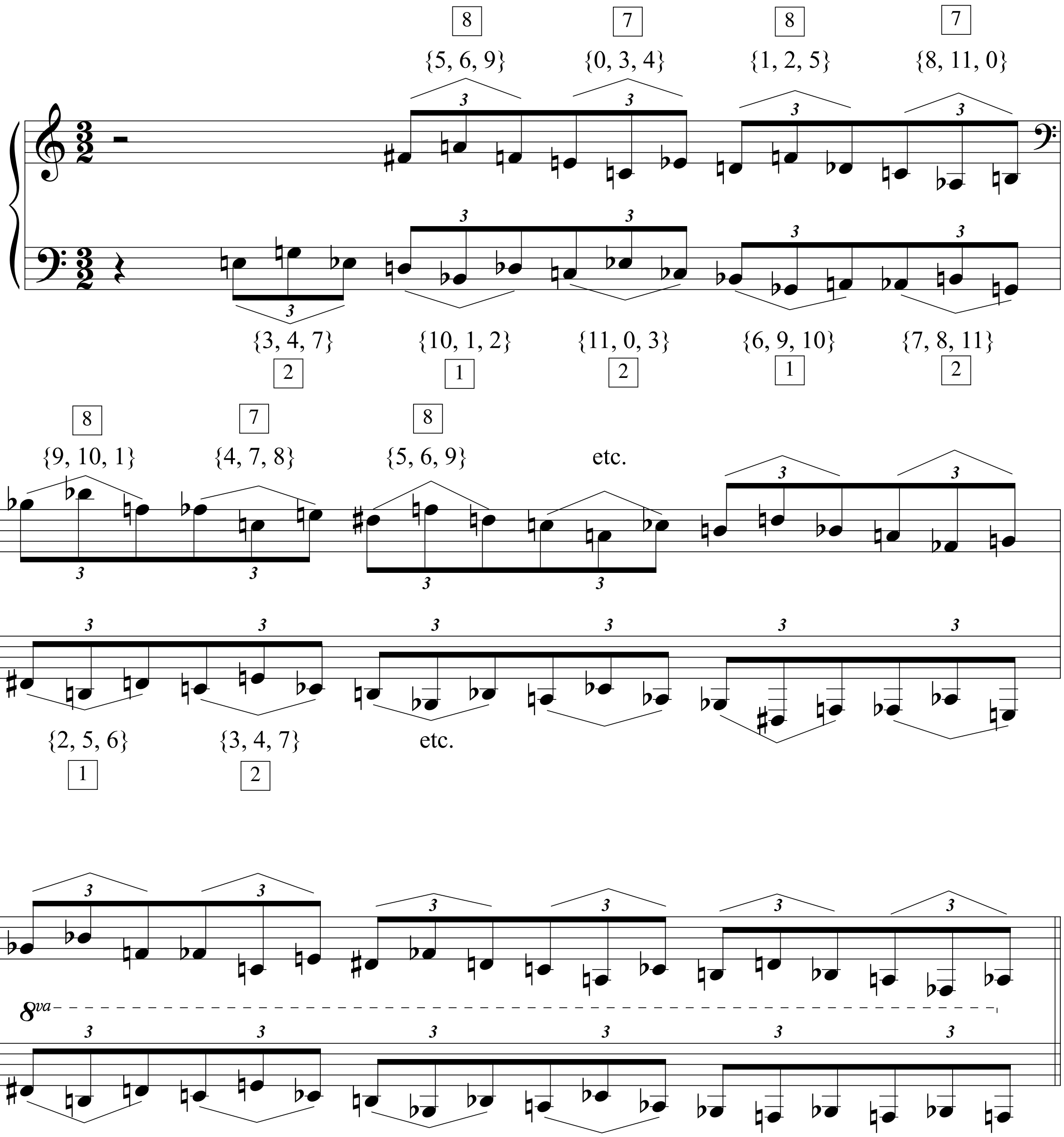
To find P~R~({0, 1, 3}), we begin by translating R~ into an I_n transformation: {0, 1, 3} is a prime form of the set class, and so Definition 3.3 tells us that R~ will be equal to I_{b+c}, which, in this case is $I_{1+3} = I_4$. I_4{0, 1, 3} = {4, 3, 1} and we then retrograde this set to {1, 3, 4} so that it is in the correct position for the next transformation. We now translate P~ into an I_n transformation via Definition 3.1, which tells us that P~ = I_{a+c} whether the set is a prime or inverted form. Thus, P~({1, 3, 4}) = $I_{1+4} = I_5$({1, 3, 4}) = {4, 2, 1}. We can see, then, that P~R~({0, 1, 3}) = {1, 2, 4} (re-ordered) and that this transformation does indeed produce the same result as T_1. Applying just P~ to {0, 1, 3} is $I_{0+3} = I_3$({0, 1, 3}) = {3, 2, 0}, which, as can be seen, inhabits a SUM class adjacent to the SUM class for {0, 1, 3}. If Δ^1 takes us from a set in [4] to a set in [7], then Δ^2 (equivalent to transposing by T_2) will take us from a set in [4] to a set in

$\boxed{10}$, Δ^3 (equivalent to transposition by T_3) from a set in $\boxed{4}$ to a set in $\boxed{1}$, Δ^4 (equivalent to transposition by T_4) from a set in $\boxed{4}$ to another set in $\boxed{4}$, and so on. Similarly, if Ω takes us from a set in $\boxed{4}$ to a set in $\boxed{5}$, then $\Delta^1\Omega$ will take us from a set in $\boxed{4}$ to a set in $\boxed{8}$, $\Delta^2\Omega$ from a set in $\boxed{4}$ to a set in $\boxed{11}$, and so on just like Δ^n but "shifted over" one SUM class. Thus, we can see that the cosets in Table 3.11 do indeed produce motion at the level of the SUM classes that is identical to the Y*n* and X*n* transformations, and so we may define an isomorphism from these cosets to the Y*n*/X*n* group as seen in Table 3.13. All that is needed to make use of the $\Delta^n/\Delta^n\Omega$ group for each set class, then, is to assign new values of P~/L~/R~ to Δ^n and Ω! For 3-3, Δ^1 = R~P~ and Ω = P~L~P~ or P~R~P~; for 3-5, Δ^1 = P~R~ and Ω = L~; and for 3-7, Δ^1 is the rather hideous L~P~R~P~ and Ω = P~.

$\Delta^n/\Delta^n\Omega$ Transformations	**Isomorphism**	**SUM-Class Transformations**
$\{\Delta^{12}, \Delta^4, \Delta^8\}$	⇔	Y0
$\{\Delta^1, \Delta^5, \Delta^9\}$	⇔	Y3
$\{\Delta^2, \Delta^6, \Delta^{10}\}$	⇔	Y6
$\{\Delta^3, \Delta^7, \Delta^{11}\}$	⇔	Y9
$\{\Delta^{12}\Omega\ (\Omega), \Delta^4\Omega, \Delta^8\Omega\}$	⇔	X1
$\{\Delta^1\Omega, \Delta^5\Omega, \Delta^9\Omega\}$	⇔	X4
$\{\Delta^2\Omega, \Delta^6\Omega, \Delta^{10}\Omega\}$	⇔	X7
$\{\Delta^3\Omega, \Delta^7\Omega, \Delta^{11}\Omega\}$	⇔	X10

Table 3.13. The homomorphism from the $\Delta^n/\Delta^n\Omega$ group onto the Y*n*/X*n* SUM-class group mediated by the isomorphism between the quotient group of the $\Delta^n/\Delta^n\Omega$ group modulo $\{\Delta^{12}, \Delta^4, \Delta^8\}$ and the Y*n*/X*n* group.

With the T_n/I_n, Zn/Wn, neo-Riemannian, and Yn/Xn groups adjusted as needed for use within set classes 3-2, 3-3, 3-5, 3-7, and 3-8, we can then create a transformational system for each of these set classes that is exactly analogous to that of the consonant triads—commutative relationships and all. That this is possible suggests that voice-leading relationships within a set class are somehow "equivalent" to voice-leading relationships within a set class that shares the same SUM-class structure. In other words, the SUM-class "profile" of a set class reveals something about its internal structure, and we may use these profiles to classify the set classes of a single cardinality into various voice-leading structures.

To see one of these systems in action, let us consider Example 3.1, which reproduces the piano part from measures 21–23 of the eighth movement of Schoenberg's *Pierrot Lunaire*. As the brackets on the example indicate, the entire expert (save the last six notes in the left hand where Schoenberg runs up against the lower limit of the piano and so cannot continue) is simply a sequence of alternating prime and inverted forms of set class 3-3. The left hand begins the sequence with a set from $\boxed{2}$ and then begins a string of $\Delta^4\Omega$ and transformations that eventually cycles through all three sets in $\boxed{1}$ and all three sets in $\boxed{2}$ before repeating an octave lower. At the level of the SUM class, the oscillation between $\boxed{2}$ and $\boxed{1}$ can be analyzed as a string of X1 transformations, which reveals that the entire sequence proceeds only by semitonal voice leading between adjacent sets. The right hand echoes this same pattern canonically but transposed the maximal six semitones (in terms of a voice-leading interval, not a T_n interval) away to $\boxed{8}$ and $\boxed{7}$. The whole sequence can thus be represented transformationally at the level of the SUM class as seen in Figure 3.1.

Example 3.1. Schoenberg, *Nacht* from *Pierrot Lunaire*, op. 21, mm. 21–23.

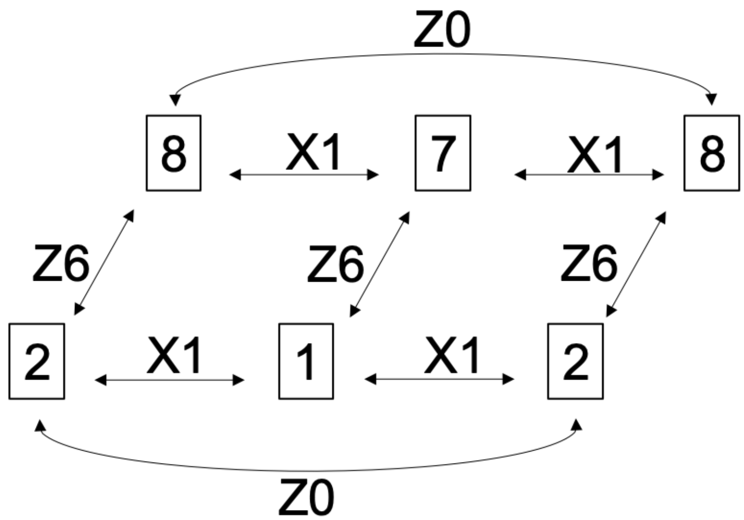


Figure 3.1. A transformational network of Schoenberg, *Nacht* from *Pierrot Lunaire*, op. 21, mm. 21–23.

Not all set classes lend themselves to these transformational systems quite so nicely, however. In the case of 3-8, for example, there can be no meaningful translation of the P~/L~/R~ transformations into the $\Delta^n/\Delta^n\Omega$ group. Applying P~, L~, and R~ to {9,1,3} in any combination and any number of times, for example, will only ever take us to sets belonging to 5 and 11, despite the fact that there are also inversionally-related sets in 2 and 5. The reason for this has to do with the fascinating way that 3-8's intervallic structure interacts with the contextual transformations. 3-8 is a whole-tone segment, and because of this, the pitch classes in each set are always separated by an even number of semitones. This causes each set to consist only of even or odd integers but never both, which is a problem for the contextual inversions because they derive their inversional axes by the sum of two pitch classes. As is well known, the sum of any two odd or any two even integers will always be an even integer, which means that a contextual inversion will only be able to produce an even inversional axis. In other words, there is no way to sum together two pitch classes from a member of 3-8 and produce an odd inversional axis, and, as a result, the contextual inversions can only move between pitch-class sets related by even inversional axes.[54] Thus, while the SUM-class structure of 3-8 nicely partitions the sets into consistent voice-leading classes, there are no currently-defined transformations that will *consistently* move between them at the level of the pitch-class set.[55]

3.2 SUM Classes for 3-1, 3-6, 3-9, and 3-10

Thus far we have observed SUM-class spaces whose structures are the same as that of the consonant triads, but this is not the only form these spaces can take. To see this, we now turn to the SUM-class systems for set classes 3-1, 3-6, 3-9, and 3-10 that are displayed in Tables 3.14 through 3.17.

SUM Class	Pitch-Class Set Members
0	{{11, 0, 1}, {3, 4, 5}, {7, 8, 9}}
3	{{0, 1, 2}, {4, 5, 6}, {8, 9, 10}}
6	{{1, 2, 3}, {5, 6, 7}, {9, 10, 11}}
9	{{10, 11, 0}, {2, 3, 4}, {6, 7, 8}}

Table 3.14. The SUM classes of set class 3-1.

[54] The members of 3-6 and 3-12 also partition themselves into only even and only odd integers, but, as we shall see, these sets do not require inversions because they are inversionally symmetrical.

[55] It is true that we could use the Z*n*/W*n* group but recall that the W*n* transformations do not produce consistent voice-leading intervals.

SUM Class	Pitch-Class Set Members
0	{{2, 4, 6}, {6, 8, 10}, {10, 0, 2}}
3	{{3, 5, 7}, {7, 9, 11}, {11, 1, 3}}
6	{{0, 2, 4}, {4, 6, 8}, {8, 10, 0}}
9	{{1, 3, 5}, {5, 7, 9}, {9, 11, 1}}

Table 3.15. The SUM classes of set class 3-6.

SUM Class	Pitch-Class Set Members
0	{{1, 3, 8}, {5, 7, 0}, {9, 11, 4}}
3	{{2, 4, 9}, {6, 8, 1}, {10, 0, 5}}
6	{{3, 5, 10}, {7, 9, 2}, {11, 1, 6}}
9	{{0, 2, 7}, {4, 6, 11}, {8, 10, 3}}

Table 3.16. The SUM classes of set class 3-9.

SUM Class	Pitch-Class Set Members
0	{{1, 4, 7}, {5, 8, 11}, {9, 0, 3}}
3	{{2, 5, 8}, {6, 9, 0}, {10, 1, 4}}
6	{{3, 6, 9}, {7, 10, 1}, {11, 2, 5}}
9	{{0, 3, 6}, {4, 7, 10}, {8, 11, 2}}

Table 3.17. The SUM classes of set class 3-10.

Each of these set classes are inversionally symmetrical about a single axis, and because of this, they contain only twelve unique members instead of twenty-four. In cases like these, there is no prime/inversion distinction in terms of the intervallic content of the sets, and, as a result, the action of transpositions and inversions are somewhat conflated with one another (we shall explore this further shortly). When these types of set classes are turned in to SUM-class spaces, the twelve unique sets partition themselves in sets of three into the four SUM classes that were not populated by the other set classes we have investigated so far: $\boxed{0}$, $\boxed{3}$, $\boxed{6}$, and $\boxed{9}$. In these spaces, then, sets in adjacent SUM classes always lie the same interval (three semitones) from one another, whereas the adjacent classes for the non-symmetrical sets could be either one or two semitones apart.

We noted a moment ago that the actions of transpositions and inversions would be somewhat conflated within these systems because there is no intervallic difference between a transposed and inverted form of a set. Indeed, within the SUM-class space for 3-1, for example, both T_1 and I_1 can be used to transform {11, 0, 1} (a member of $\boxed{0}$) into {0, 1, 2} (a member of $\boxed{3}$). However, this does not mean that the behavior of these two transformations is identical. Not only do T_1 and I_1 only conflate when applied to {11, 0, 1},[56] but I_1—as with all inversional transformations—is also an involution, meaning that it is its own inverse. Applying I_1 to {0, 1, 2} will thus take us back to {11, 0, 1}, whereas T_1 will continue "in the same direction" when applied to {0, 1, 2} as it did when applied to {11, 0, 1} and take us to {1, 2, 3} (a member of $\boxed{6}$). While there is no difference between a transposed and inverted form of 3-1 (or any of the other

[56] T_1 of {3, 4, 5} (another member of 3-1 $\boxed{0}$) is {4, 5, 6} whereas I_1 is {8, 9, 10} (both members of 3-1 $\boxed{3}$, but notably not the same set).

set classes with the same structure) in terms of intervallic content, then, there *is* a difference in the way that sets are affected by transpositions and inversions: sets in $\boxed{0}$ and $\boxed{6}$ are always sent in the same direction by transpositions and inversions whereas sets in $\boxed{3}$ and $\boxed{9}$ are always sent in the opposite direction.

Even though the T_n and I_n transformations are broadly the same in these spaces, then, they cannot be placed into the same SUM-class transformations because this would mean that the same SUM-class transformation would do two different things when applied to the very same SUM class. But because these sets are inversionally symmetrical, there is no need to invoke both the transpositional and the inversional transformations since the transpositions by themselves are able to account for the transformation of any member of each set class to any other member of the same set class. Indeed, the transformations on their own actually form a proper group! The group identity is T_0, the inverse of any T_n is T_{12-n}, the product of any two transposition will always be another transposition, and these transformations will always be associative since they operate based on simple addition. Furthermore, $\{T_0, T_4, T_8\}$ is still a normal subgroup within this smaller group, and the cosets of $\{T_0, T_4, T_8\}$ within the T_n will be the same as the transposition cosets of the T_n/I_n group: $\{T_0, T_4, T_8\}$, $\{T_1, T_5, T_9\}$, $\{T_2, T_6, T_{10}\}$, and $\{T_3, T_7, T_{11}\}$. When the T_n group is applied to the space of any of the symmetrical trichord classes, we find that these cosets always produce the same mappings at the SUM-class level. In the case of 3-1, for example, it can easily be seen that T_0, T_4, and T_8 will always map between sets in the same SUM class while $\{T_1, T_5, T_9\}$, $\{T_2, T_6, T_{10}\}$, and $\{T_3, T_7, T_{11}\}$ will map between sets in SUM-classes that are three, six, and nine semitones away respectively.

It should not surprise us, then, that the *Zn* transformations by themselves can also form a group of their own and that this group is isomorphic to the T_n quotient group. This isomorphism is shown in Table 3.18, and the actions of the *Zn* group on $\boxed{0}$, $\boxed{3}$, $\boxed{6}$, and $\boxed{9}$ are shown in Table 3.19. The GISs created from these groups and their respective space are much simpler than for the non-symmetrical sets we observed above because they are commutative, and commutative groups do not necessarily have duals in the same way that non-commutative groups do. Even if there were to be another group of transformations that could act upon these spaces, however, these two GISs generalize voice leading so well that there is no need for any others for the context of this thesis. As a whole, this system for the symmetrical trichords could be depicted visually as the left half (GIS^1 and GIS^3) of Figure 2.1 (from Chapter 2).

T_n Transformations	**Isomorphism**	***Zn* Transformations**
$\{T_0, T_4, T_8\}$	⇔	Z0
$\{T_1, T_5, T_9\}$	⇔	Z3
$\{T_2, T_6, T_{10}\}$	⇔	Z6
$\{T_3, T_7, T_{11}\}$	⇔	Z9

Table 3.18. The homomorphism from the T_n group onto the *Zn* group mediated by the isomorphism between the quotient group of the T_n group modulo $\{T_0, T_4, T_8\}$ and the *Zn* group.

Zn **SUM-class Transformations**	**Action on Sum Classes $\boxed{0}$, $\boxed{3}$, $\boxed{6}$, $\boxed{9}$**
Z0	($\boxed{0}$) ($\boxed{3}$) ($\boxed{6}$) ($\boxed{9}$)
Z3	($\boxed{0}$, $\boxed{3}$, $\boxed{6}$, $\boxed{9}$)
Z6	($\boxed{0}$, $\boxed{6}$) ($\boxed{3}$, $\boxed{9}$)
Z9	($\boxed{0}$, $\boxed{9}$, $\boxed{6}$, $\boxed{3}$)

Table 3.19. The actions of the Z*n* transformations on $\boxed{0}$, $\boxed{3}$, $\boxed{6}$, and $\boxed{9}$.

3.3 SUM Classes for 3-4 and 3-12

Set class 3-12 (the augmented triads) exhibits the same SUM-class structure as the other symmetrical sets we explored above, but its three degrees of inversional symmetry leave only four unique triads, each of which inhabits its own SUM class (See Table 3.20). Even with only one triad in each class, however, the $\{T_0, T_4, T_8\}$ cosets will still produce the same mappings at the SUM-class level because the actions of these transformations on any augmented triad are identical. For example, $T_1(\{0, 4, 8\}) = \{1, 5, 9\}$, $T_5(\{0, 4, 8\}) = \{5, 9, 1\}$, and $T_9(\{0, 4, 8\}) = \{9, 1, 5\}$. Trivial though this space is, then, its structure is the same as the spaces of the other symmetrical set classes.

SUM Class	Pitch-Class Set Members
0	{0, 4, 8}
3	{1, 5, 9}
6	{2, 6, 10}
9	{3, 7, 11}

Table 3.20. The SUM classes of set class 3-12.

The SUM-class structure of 3-4 (seen in Table 3.21) is unique among the trichordal set classes. Like the symmetrical set classes, the members of 3-4 reside in 0, 3, 6, and 9, but because 3-4 is *not* a symmetrical set class, it generates twenty-four unique sets rather than twelve.[57] These four SUM classes thus hold twice as many sets as the other SUM-class spaces we have seen so far. More significantly, however, each SUM class contains both prime and inverted forms of the set class. In order to move through this space, then, we will need SUM-class transformations that are isomorphic to cosets containing both transpositions and inversions. But we already noted earlier that a single SUM-class transformation cannot contain both transpositions and inversions because this would cause the transformation in question to behave inconsistently on the SUM classes.

[57] Straus calls these sets "mavericks." See Straus, "Sum class," 9.

SUM Class	Pitch-Class Set Members
0	{{2, 3, 7}, {9, 1, 2}, {6, 7, 11}, {1, 5, 6}, {10, 11, 3}, {5, 9, 10}}
3	{{3, 4, 8}, {10, 2, 3}, {7, 8, 0}, {2, 6, 7}, {11, 0, 4}, {6, 10, 11}}
6	{{0, 1, 5}, {7, 11, 0}, {4, 5, 9}, {11, 3, 4}, {8, 9, 1}, {3, 7, 8}}
9	{{1, 2, 6}, {8, 0, 1}, {5, 6, 10}, {0, 4, 5}, {9, 10, 2}, {4, 8, 9}}

Table 3.21. The SUM classes of set class 3-4.

To work around this problem, let us re-define the equivalence relation that created this SUM-class space so that it segregates the prime and inverted forms of the set class into separate spaces:

Definition 3.4. Let *S* be a set of all pitch-class sets in set class 3-4 and *R* a relation on *S* such that $(s, t) \in R$ if an only if SUM(s) = SUM(t) and $t = T_n(s)$ for some $n = 0, 1, \ldots 11$.

That is, two pitch-class sets will belong to the same SUM class if they have the same SUM value *and* can be transformed into one another via some sort of transposition (meaning that they are either both primes or both inversions). To prove that this is an equivalence relation, we must only check the second condition ($t = T_n(s)$) since we already know that the first condition guarantees an equivalence relation.

Proof: (s, s) will always be in *R* because $T_0(s) = s$. If $(s, t) \in R$ then (t, s) will also be in *R* because if $T_n(s) = t$ then $T_n^{-1}(t) = s$ and the inverse of every *n* is a possible value of T_n since we are dealing with mod-12 arithmetic. Finally, if (s, t) and $(t, u) \in R$, (s, u) will also be in *R* because if $T_m(s) = t$ and $T_n(t) = u$ then $T_{m+n}(s) = u$, and since T_n is a closed group under addition, every value of T_{m+n} will also be a possible value of T_n.

Defined this way, the equivalence relation creates two different SUM-class spaces from the pitch-class sets in set class 3-4, which can be seen below in Tables 3.22 and 3.23. With the primes and inversions segregated into different spaces, we can then make use of the exact same groups as we did for the other non-symmetrical trichords: T_n/I_n, whose quotient group modulo $\{T_0, T_4, T_8\}$ is isomorphic to the Zn/Wn group and $\Delta^n/\Delta^n\Omega$ with Δ^1 = P~R~ and Ω = P~R~P~, whose quotient group modulo $\{\Delta^{12}, \Delta^4, \Delta^8\}$ is isomorphic to the Yn/Xn group. The Yn and Xn transformations were originally defined by Cohn to only act on SUM class congruent to 1 and 2 modulo 3, however, and so to use them here for SUM classes congruent to 0 mod 3 we must redefine them slightly as seen in Definitions 3.5 and 3.6. Once redefined in this way, Table 3.24 reveals that the group will once again act simply transitively upon the SUM-class space, meaning that we can create GISs from these transformations. From a transformational perspective, then, the SUM-class space for 3-4 is essentially analogous to the spaces for the other non-symmetrical trichords. In regard the actual voice leading within the set class, however, 3-4 is much more like the symmetrical trichords. Thus, we can say that 3-4 is something of a hybrid of the two possible trichordal SUM-class profiles.

Definition 3.5. $Yn(s) = s + n$; $Yn(s') = s' - n$.

Definition 3.6. $Xn(s) = s' + n$; $Xn(s') = s - n$.

SUM Class	Pitch-Class Set Members
0	{{2, 3, 7}, {6, 7, 11}, {10, 11, 3}}
3	{{3, 4, 8}, {7, 8, 0}, {11, 0, 4}}
6	{{0, 1, 5}, {4, 5, 9}, {8, 9, 1}}
9	{{1, 2, 6}, {5, 6, 10}, {9, 10, 2}}

Table 3.22. The SUM classes for the prime forms of set class 3-4.

SUM Class	Pitch-Class Set Members
0’	{{9, 1, 2}, {1, 5, 6}, {5, 9, 10}}
3’	{{10, 2, 3}, {2, 6, 7}, {6, 10, 11}}
6’	{{7, 11, 0}, {11, 3, 4}, {3, 7, 8}}
9’	{{8, 0, 1}, {0, 4, 5}, {4, 8, 9}}

Table 3.23. The SUM classes for the inverted forms of set class 3-4.

Y*n*/X*n* SUM-Class Transformations	**Action on the Sum Classes of 3-4**
Y0	(0) (0’) (3) (3’) (6) (6’) (9) (9’)
Y3	(0, 3, 6, 9) (0’, 9’, 6’, 3’)
Y6	(0, 6) (3, 9) (0’, 6’) (3’, 9’)
Y9	(0, 9, 6, 3) (0’, 3’, 6’, 9’)
X0	(0, 0’) (3, 3’) (6, 6’) (9, 9’)
X3	(0, 3’) (3, 6’) (6, 9’) (9, 0’)
X6	(0, 6’) (3, 9’) (6, 0’) (9, 3’)
X9	(0, 9’) (3, 0’) (6, 3’) (9, 6’)

Table 3.24. The actions of the redefined Y*n*/X*n* group on the SUM classes of 3-4.

3.4 SUM Classes Generalized for All Trichords

We have now constructed SUM-class spaces for all of the trichords and have observed two basic forms that these spaces may take. The question, then, is what determines which type of SUM-class space a particular set class generates. If it were not for the exceptional case of 3-4, it would appear that SUM-class space is solely determined by whether or not a set class is inversionally symmetrical, but this set class obviously problematizes that notion. To begin investigating this, we must first remember that a set class is a set of pitch-class sets that are all related to one another via some T_n and/or some I_n transformation. In other words, it would be possible to generate an entire set class by simply performing every T_n and I_n transformation on a single set. For example, applying the twelve T_n and twelve I_n transformations to a C + would return twelve unique major and twelve unique minor triads (including C+ again). These twenty-four triads are thus members of the same set class (which Forte labels as 3-11).

To understand how a set class will generate a SUM-class space, then, we must understand the way that the T_n and I_n transformations interact with the SUM classes. To do this, let us consider an abstract trichord, S that contains the abstract pitch classes a, b, c. Recall that T_n operates by adding n (mod 12) to each pitch class in the pitch-class set it is applied to. $T_n(S)$ is thus equal to $T_n(\{a, b, c\}) = \{(n + a), (n + b), (n + c)\}$. The SUM function (defined in Chapter 1) acts on a pitch-class set by summing (mod 12) all of the pitch classes in a given pitch-class set. SUM of S, then, is equal to $(a + b + c)$, and so S will belong to $\boxed{a + b + c}$—the equivalence class of all pitch-class sets with the same SUM value. The results of $T_n(S)$ and SUM(S) can be interrelated to one another: $\mathrm{SUM}(T_n(S)) = \mathrm{SUM}(\{(n + a), (n + b), (n + c)\}) = ((n + a) + (n + b) + (n + c))$, which can then be simplified to $3n + (a + b + c)$. With this little equality, we can then know that T_n of *any* set in $\boxed{X}$ will always belong to $\boxed{3n + X}$.

The actions of I_n and SUM can also be interrelated. The I_n transformation acts on a set by subtracting each pitch-class in a given set from the variable n: $I_n(S) = I_n(\{a, b, c\}) = \{(n - a), (n - b), (n - c)\}$. $\mathrm{SUM}(I_n(S))$, then, is equal to $\mathrm{SUM}(\{(n - a), (n - b), (n - c)\}) = ((n - a) + (n - b) + (n - c))$, which can be simplified to $3n - (a + b + c)$. The I_n transform of *any* set in $\boxed{X}$ will thus always be a set in $\boxed{3n - X}$.

We now have two equations that describe the interaction of T_n and I_n with the SUM classes: $T_n(x \in \boxed{X}) = x \in \boxed{3n + X}$ and $I_n(x \in \boxed{X}) = x \in \boxed{3n - X}$. One of the things these equations can tell us is that trichords related by transposition will always belong to SUM classes with the same value modulo 3 whereas trichords related by inversion will always belong to SUM classes whose values will sum to 0 mod 3. Furthermore, solving 3*n* for every *n* from 0 to 11 will only ever produce four unique values: 0, 3, 6, 9. Thus, and T_n-related sets will always be in SUM classes that are either zero, three, six, or nine semitones apart and I_n-related sets will always belong to SUM classes that are inversions of one another about pitch-class axes 0, 3, 6, or 9. This should sound familiar to us, because this is exactly how the Z*n* and W*n* transformations are defined to act upon the SUM classes. Indeed, the above equations are just another way of showing this relationship between T_n, I_n, and the SUM classes: all T_n transformations for which 3*n* is the same will belong to the same Z*n* transformation and likewise for the I_n and W*n* transformations. Knowing this, it is then possible to make the following generalizations about the SUM-class structure of a trichordal set class given only the SUM value of the set class's prime-form representative:

1) If SUM of the prime-form representative is congruent to 0 mod 3, all members of the set class will inhabit SUM classes [0], [3], [6], or [9] since two integers congruent to 0 mod 3 will also sum to 0 mod 3.

2) If SUM of the prime-form representative is congruent to 1 mod 3, all prime forms of the set class will inhabit SUM classes [1], [4], [7], or [10] and the inverted forms will inhabit SUM classes [2], [5], [8], and [11].

3) If SUM of the prime-form representative is congruent to 2 mod 3, all prime forms of the set class will inhabit SUM classes [2], [5], [8], or [11] and the inverted forms will inhabit SUM classes [1], [4], [7], and [10].

In the first case where all members of the set class inhabit SUM classes [0], [3], [6], and [9], the sets in these SUM classes will be related to one another by transposition *and* inversion. This does not necessarily mean that sets in these classes need be inversionally symmetrical (as we saw with 3-4), but it does mean that inversionally-symmetrical set classes can *only* inhabit these SUM classes. If a symmetrical set were to SUM to 1 mod 3, for example, then all sets related to this set would be placed into SUM classes $\boxed{2}$, $\boxed{5}$, $\boxed{8}$, and $\boxed{11}$. But since this is a symmetrical set class, these sets would also be transpositionally related to the sets in $\boxed{1}$, $\boxed{4}$, $\boxed{7}$, and $\boxed{10}$ and we already saw above that sets related by transposition will always belong to sets that are congruent to one another mod 3. Such a situation would thus create a contradiction.

While it is true that none of the above generalizations explain why 3-4 is the only non-symmetrical set class to inhabit SUM classes $\boxed{0}$, $\boxed{3}$, $\boxed{6}$, and $\boxed{9}$, these generalizations do at least indicate that such a situation would be possible. We shall continue to examine similar exceptional cases as we encounter them in other cardinalities, but for the moment it appears that 3-4 just happens to be the only non-symmetrical set class whose pitch classes sum to 0 mod 3.

3.5 SUM Classes for Tetrachords

There is nothing about any of the ideas we have explored so far in this thesis that are necessarily cardinality specific. Indeed, the two functions that are fundamental to the SUM-class idea—SUM and PVLS—are defined so as to be cardinality agnostic. As long as larger sets continue to display the same relationship between SUM and PVLS, then, there is no reason why everything we have studied so far might not be extended to larger-cardinality sets as well. We begin here with four-note sets.

The two equations we developed in the previous section to generalize the possible three-note SUM-class profiles can also help us explore the possible SUM-class profiles for four-note sets. Since T_n and/or I_n will now be acting on sets containing four pitch classes, however, our equations will need to be modified slightly: $T_n(x \in \boxed{X}) = x \in \boxed{4n + X}$ and $I_n(x \in \boxed{X}) = x \in \boxed{4n - X}$. Sets related to one another by transposition will now belong to set classes congruent to one another mod 4 and inversionally-related sets will belong to SUM classes that sum to 0 mod four. We then have four possible scenarios that might describe a tetrachord's SUM-class profile:

1) If SUM of the prime-form representative is congruent to 0 mod 4, all members of the set class will inhabit [0], [4], or [8].

2) If SUM of the prime-form representative is congruent to 1 mod 4, all prime forms of the set class will inhabit [1], [5], or [9] and all inverted forms of the set class will inhabit [3], [7], or [11].

3) If SUM of the prime-form representative is congruent to 2 mod 4, all members of the set class will inhabit [2], [6], or [10].

4) If SUM of the prime-form representative is congruent to 3 mod 4, all prime forms of the set class will inhabit [3], [7], or [11] and the inverted forms of the set class will inhabit [1], [5], or [9].

From our discussion of the trichords it should be apparent that inversionally-symmetrical trichords will only be able to inhabit the [0], [4], [8] and [2], [6], [10] spaces while the non-symmetrical set classes might inhabit any of the three spaces. As we saw with the trichords, set classes sharing the same SUM-class profile have the same characteristics in general, and so we shall only examine a few representatives from each of the three broad profiles and any exceptional cases, but a complete list of the SUM-class profiles exhibited by each set class can be seen in Table 3.25.

0, 4, 8	2, 6, 10	1, 3, 5, 7, 9, 11
4-3*	4-1*	4-2
4-4	4-6*	4-5
4-8*	4-7*	4-11
4-14	4-9*	4-12
4-18	4-10*	4-z15
4-21*	4-13	4-16
4-25*	4-17*	4-19
4-26*	4-20*	4-22
	4-23*	4-27
	4-24*	4-z29
	4-28*	

Table 3.25. The tetrachordal set classes that exhibit each of the three possible tetrachordal SUM-class profiles. An asterisk (*) indicates that the set class is inversionally symmetrical.

We begin with the ten set classes inhabiting {1, 3, 5, 7, 9, 11} (the "whole-tone" SUM classes). Each of these set classes is non-symmetric and its twelve unique primes and twelve unique inversions are nicely partitioned in sets of four into separate SUM classes. Let us consider the SUM-class system of 4-27 (the dominant and half-diminished seventh chords) seen below in Table 3.26.[58]

[58] This space is also discussed briefly in Cohn, "Square Dances with Cubes," 295.

SUM Classes	Pitch-Class Set Members
1	{{5, 8, 11, 1}, {8, 11, 2, 4}, {11, 2, 5, 7}, {2, 5, 8, 10}}
3	{{0, 2, 5, 8}, {3, 5, 8, 11}, {6, 8, 11, 2}, {9, 11, 2, 5}}
5	{{6, 9, 0, 2}, {9, 0, 3, 5}, {0, 3, 6, 8}, {3, 6, 9, 11}}
7	{{1, 3, 6, 9}, {4, 6, 9, 0}, {7, 9, 0, 3}, {10, 0, 3, 6}}
9	{{4, 7, 10, 0}, {7, 10, 1, 3}, {10, 1, 4, 6}, {1, 4, 7, 9}}
11	{{2, 4, 7, 10}, {5, 7, 10, 1}, {8, 10, 1, 4}, {11, 1, 4, 7}}

Table 3.26. The SUM classes of set class 4-27.

Here, the dominant seventh chords (the inverted forms of the set class) occupy SUM classes 1, 5, and 9 and the half-diminished sevenths occupy classes 3, 7, and 11. As with the trichords, the four tetrachords in each SUM class lie a voice-leading interval of zero semitones away from one another, and the voice leading between all members of one SUM class and all members of another will be the same. It should be clear, however, that the SUM-class transformations we used for the trichords will not be applicable here. Y3, for instance, does not map any of the SUM classes seen in Table 3.26 to a SUM class that is also in the same space: Y3(1) = 4, Y3(3) = 6, etc. This does not mean that the Y*n*/X*n* transformations will not work for four-note sets, just that this *particular* group of transformations will not. But the SUM-class transformations that we have used so far are actually only one of many possible subgroups that can be extracted from the full order-twenty-four Y*n*/X*n* group: {Y0, Y1, Y2, Y3, Y4, Y5, Y6, Y7, Y8, Y9, Y10, Y11, X0, X1, X2, X3, X4, X5, X6, X7, X8, X9, X10, X11}. Because the space we are trying to act upon is of cardinality six, only an order-six subgroup will act simply transitively upon the space. There are five order-six subgroups of the Y*n*/X*n* group: {Y0, Y2, Y4, Y6, Y8, Y10}, {Y0, Y4, Y8, X0, X4, X8}, {Y0, Y4, Y8, X1, X5, X9}, {Y0, Y4, Y8, X2, X6, X10}, and {Y0, Y4, Y8, X3, X7, X11}. Of these, only {Y0, Y4, Y8, X0, X4, X8} acts simply transitively upon the {$\boxed{1}$, $\boxed{3}$, $\boxed{5}$, $\boxed{7}$, $\boxed{9}$, $\boxed{11}$} space when defined as in Definition 3.7. Table 3.27 displays these mappings.

Definition 3.7. TR$n(s) = s + n$ (mod 12) if $s \equiv 1$ mod 4 or $s - n$ (mod 12) if $s \equiv 3$ mod 4.

Y*n*/X*n* SUM-Class Transformations	**Action on SUM classes**
Y0	($\boxed{1}$) ($\boxed{3}$) ($\boxed{5}$) ($\boxed{7}$) ($\boxed{9}$) ($\boxed{11}$)
Y4	($\boxed{1}$, $\boxed{5}$, $\boxed{9}$) ($\boxed{3}$, $\boxed{11}$, $\boxed{7}$)
Y8	($\boxed{1}$, $\boxed{9}$, $\boxed{5}$) ($\boxed{3}$, $\boxed{7}$, $\boxed{11}$)
X2	($\boxed{1}$, $\boxed{3}$) ($\boxed{5}$, $\boxed{7}$) ($\boxed{9}$, $\boxed{11}$)
X6	($\boxed{1}$, $\boxed{7}$) ($\boxed{3}$, $\boxed{9}$) ($\boxed{5}$, $\boxed{11}$)
X10	($\boxed{1}$, $\boxed{11}$) ($\boxed{3}$, $\boxed{5}$) ($\boxed{7}$, $\boxed{9}$)

Table 3.27. The actions of the {Y0, Y4, Y8, X2, X6, X10} subgroup on the SUM classes.

Aside from the fact that we are using different transformations and acting on a slightly different space, the essential structure of this group is the same as the Y*n*/X*n* subgroup we used for the trichords: Y0 is the identity, each X*n* is its own inverse, Y*n* transformations whose indexes sum to 0 mod 12 are each other's inverses, and the group is associative and closed under the same binary composition as the trichordal Y*n*/X*n* group (this is left to the reader to check). The group is also non commutative, which means it also has a dual Z*n*/W*n* group to which it is isomorphic. Once again, the trichordal Z*n*/W*n* group will not work here, but that group was also merely a subgroup of an order-twenty-four group of twelve Z*n* and twelve W*n* transformations. The order-six Z*n*/W*n* subgroup that does act on this space can be seen in Table 3.28.

***Zn*/W*n* SUM-Class Transformations**	**Action on SUM classes**
Z0	($\boxed{1}$) ($\boxed{3}$) ($\boxed{5}$) ($\boxed{7}$) ($\boxed{9}$) ($\boxed{11}$)
Z4	($\boxed{1}$, $\boxed{5}$, $\boxed{9}$) ($\boxed{3}$, $\boxed{7}$, $\boxed{11}$)
Z8	($\boxed{1}$, $\boxed{9}$, $\boxed{5}$) ($\boxed{3}$, $\boxed{11}$, $\boxed{7}$)
W0	($\boxed{1}$, $\boxed{11}$) ($\boxed{3}$, $\boxed{9}$) ($\boxed{5}$, $\boxed{7}$)
W4	($\boxed{1}$, $\boxed{3}$) ($\boxed{5}$, $\boxed{11}$) ($\boxed{7}$, $\boxed{9}$)
W8	($\boxed{1}$, $\boxed{7}$) ($\boxed{3}$, $\boxed{5}$) ($\boxed{9}$, $\boxed{11}$)

Table 3.28. The actions of the {Z0, Z4, Z8, W0, W4, W8} subgroup on the SUM classes.

As can be seen from Tables 3.27 and 3.28, both the Y*n*/X*n* and Z*n*/W*n* groups act simply transitively upon the {$\boxed{1}$, $\boxed{3}$, $\boxed{5}$, $\boxed{7}$, $\boxed{9}$, $\boxed{11}$} space and as such we can once again generate two "dual" GISs that share the same space and whose transformations commute with one another. In short, though both of these GISs are superficially different from their trichordal counterparts, they are essentially analogous from a functional perspective.

The $\Delta^n/\Delta^n\Omega$ and T_n/I_n quotient groups we used for the trichords were generated from an order-three normal subgroup that acted as a miniature simply-transitive group upon the three pitch-class sets contained within each of the eight SUM classes. By analogy, then, the $\Delta^n/\Delta^n\Omega$ and T_n/I_n quotient groups for four-note sets will need to be generated from an *order-four* subgroup that will act simply-transitively upon the four pitch-class sets within each SUM class. There are seven possible order-four subgroups of the T_n/I_n group: $\{T_0, T_3, T_6, T_9\}$, $\{T_0, T_6, I_0, I_6\}$, $\{T_0, T_6, I_1, I_7\}$, $\{T_0, T_6, I_2, I_8\}$, $\{T_0, T_6, I_3, I_9\}$, $\{T_0, T_6, I_4, I_{10}\}$, and $\{T_0, T_6, I_5, I_{11}\}$. Only $\{T_0, T_3, T_6, T_9\}$ acts simply transitively upon the sets in the SUM classes of Table 3.26, however. From this normal subgroup, we generate the six cosets seen in Table 3.29.

$\{T_0, T_3, T_6, T_9\}$

$\{T_1, T_4, T_7, T_{10}\}$

$\{T_2, T_5, T_8, T_{11}\}$

$\{I_0, I_3, I_6, I_9\}$

$\{I_1, I_4, I_7, I_{10}\}$

$\{I_2, I_5, I_8, I_{11}\}$

Table 3.29. The cosets of $\{T_0, T_3, T_6, T_9\}$ in the T_n/I_n group.

As one might begin to expect by now, there is a homomorphism from the T_n/I_n group onto the Zn/Wn subgroup we defined above through exactly these cosets. This homomorphism is displayed in Table 3.30. Clearly, we could also generate the homomorphism from $\Delta^n/\Delta^n\Omega$ onto Yn/Xn by simply copying Table 3.30 and substituting $\Delta^n/\Delta^n\Omega$ for T_n/I_n and Yn/Xn for Zn/Wn.

T_n/I_n Transformations	Isomorphism	Zn/Wn Transformations
$\{T_0, T_3, T_6, T_9\}$	⇔	Z0
$\{T_1, T_4, T_7, T_{10}\}$	⇔	Z4
$\{T_2, T_5, T_8, T_{11}\}$	⇔	Z8
$\{I_0, I_3, I_6, I_9\}$	⇔	W0
$\{I_1, I_4, I_7, I_{10}\}$	⇔	W4
$\{I_2, I_5, I_8, I_{11}\}$	⇔	W8

Table 3.30. The homomorphism from the T_n/I_n group onto the order-six *Zn*/W*n* subgroup for the tetrachords mediated by the isomorphism between the quotient group of the T_n/I_n group modulo $\{T_0, T_3, T_6, T_9\}$ and the order-six *Zn*/W*n* subgroup for the tetrachords.

In order to use the $\Delta^n/\Delta^n\Omega$ group, however, we must first adapt Straus's P~, L~, and R~ transformations so that they are defined to act upon four-note sets[59]:

Definition 3.8. P~($\langle a, b, c, d\rangle$) = I_{a+d}.

Definition 3.9. L~($\langle a, b, c, d\rangle$) = I_{a+b} if the set is prime or I_{c+d} if the set is inverted.

Definition 3.10. R~($\langle a, b, c, d\rangle$) = I_{a+c} if the set is prime or I_{b+d} if the set is inverted.

These transformations alone will not be enough to move through the SUM classes of 4-27, however, because P~, L~, and R~ all move to the same SUM classes. For example, P~({5, 8, 11, 1}) = {5 ,7 , 10, 1}; L~({5, 8, 11, 1}) = {11, 1, 4, 7}; R~({5, 8, 11, 1}) = {8, 10, 1, 4}, all of which belong to $\boxed{11}$. This is because, somewhat like we saw with 3-8, the relationship between pairs of pitch classes in these sets is such that the sums obtained from P~, L~, and R~ always create inversional axes that move to sets in the same SUM class. Luckily for us, Straus also

[59] Adapted from Straus, "Contextual-Inversion Spaces," 65.

defines three more contextual inversions—P’, L’, and R’, see definitions below—that happen to move to sets in different SUM classes within the 4-27 space.[60] With Δ^1 = R~P~R~L’ and Ω = L’, we can then use the $\Delta^n/\Delta^n\Omega$ group on 4-27.

Definition 3.11. P’($<a, b, c, d>$) = I_{b+c}.

Definition 3.12. L’($<a, b, c, d>$) = I_{c+d} if the set is prime or I_{a+b} if the set is inverted.

Definition 3.13. R’($<a, b, c, d>$) = I_{b+d} if the set is prime or I_{a+c} if the set is inverted.

We have now successfully translated the generalized voice-leading system for the non-symmetrical trichords into an exactly-analogous system for all of the tetrachords that inhabit {$\boxed{1}$, $\boxed{3}$, $\boxed{5}$, $\boxed{7}$, $\boxed{9}$, $\boxed{11}$}. As before, this system allows us to transformationally generalize all voice-leading intervals between the pitch-class sets of the same set class. Among some of the more interesting set classes to which this system is applicable (in addition to 4-27 as seen above) are the two z-related tetrachords (4-z15 and 4-z29). The SUM-class spaces for these set classes are displayed in Tables 3.31 and 3.32. To make use of the $\Delta^n/\Delta^n\Omega$ group for these set classes, we assign Δ^1 = P~L~ and Ω = P~L~P~ for 4-z15 and Δ^1 = P’R~ and Ω = L~R~L~ for 4-z29.

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[60] Adapted from Straus, “Contextual-Inversion Spaces,” 65.

SUM Classes	Pitch-Class Set Members
1	{{6, 8, 11, 0}, {9, 11, 2, 3}, {0, 2, 5, 6}, {3, 5, 8, 9}}
3	{{1, 2, 5, 7}, {4, 5, 8, 10}, {7, 8, 11, 1}, {10, 11, 2, 4}}
5	{{7, 9, 0, 1}, {10, 0, 3, 4}, {1, 3, 6, 7}, {4, 6, 9, 10}}
7	{{2, 3, 6, 8}, {5, 6, 9, 11}, {8, 9, 0, 2}, {11, 0, 3, 5}}
9	{{8, 10, 1, 2}, {11, 1, 4, 5}, {2, 4, 7, 8}, {5, 7, 10, 11}}
11	{{0, 1, 4, 6}, {3, 4, 7, 9}, {6, 7, 10, 0}, {9, 10, 1, 3}}

Table 3.31. The SUM classes of set class 4-z15.

SUM Classes	Pitch-Class Set Members
1	{{5, 9, 11, 0}, {8, 0, 2, 3}, {11, 3, 5, 6}, {2, 6, 8, 9}}
3	{{1, 2, 4, 8}, {4, 5, 7, 11}, {7, 8, 10, 2}, {10, 11, 1, 5}}
5	{{6, 10, 0, 1}, {9, 1, 3, 4}, {0, 4, 6, 7}, {3, 7, 9, 10}}
7	{{2, 3, 5, 9}, {5, 6, 8, 0}, {8, 9, 11, 3}, {11, 0, 2, 6}}
9	{{7, 11, 1, 2}, {10, 2, 4, 5}, {1, 5, 7, 8}, {4, 8, 10, 11}}
11	{{0, 1, 3, 7}, {3, 4, 6, 10}, {6, 7, 9, 1}, {9, 10, 0, 4}}

Table 3.32. The SUM classes of set class 4-z29.

The remainder of the tetrachordal set classes inhabit either {0, 4, 8} or {2, 6, 10} (the "augmented" SUM classes). The majority of these classes are inversionally symmetric about a single axis (producing only twelve unique forms) and thus contain four pitch-class sets in each of their three SUM classes. Examples of this type of system are the minor and major seventh chords (4-20 and 4-26) seen in Tables 3.33 and 3.34. Sets with more than one axis of inversional symmetry (4-9, 4-25, and 4-28) also possess this same SUM-class structure but with even fewer sets in each class. In both cases, like the systems for the inversionally-symmetric trichordal classes, there is no prime/inversion distinction, and so there is no need to make use of any inversional transformations. The three Z*n* transformations from the tetrachordal Z*n*/W*n* subgroup we saw above also form their own subgroup, and this subgroup acts simply transitively upon both the {$\boxed{0}$, $\boxed{4}$, $\boxed{8}$} and {$\boxed{2}$, $\boxed{6}$, $\boxed{10}$} spaces. Similarly, the cosets of the T_n/I_n quotient group containing just T_n transformations also form their own subgroup that acts simply transitively upon the pitch-class sets in each set class. Thus, the two groups seen in Table 3.35 are the only transformations needed for any of the symmetrical tetrachords.

SUM Classes	**Pitch-Class Set Members**
$\boxed{2}$	{{0, 1, 5, 8}, {3, 4, 8, 11}, {6, 7, 11, 2}, {9, 10, 2, 5}}
$\boxed{6}$	{{1, 2, 6, 9}, {4, 5, 9, 0}, {7, 8, 0, 3}, {10, 11, 3, 6}}
$\boxed{10}$	{{11, 0, 4, 7}, {2, 3, 7, 10}, {5, 6, 10, 1}, {8, 9, 1, 4}}

Table 3.33. The SUM classes of set class 4-20.

SUM Classes	**Pitch-Class Set Members**
$\boxed{0}$	{{5, 8, 10, 1}, {2, 5, 7, 10}, {8, 11, 1, 4}, {11, 2, 4, 7}}
$\boxed{4}$	{{0, 3, 5, 8}, {6, 9, 11, 2}, {3, 6, 8, 11}, {9, 0, 2, 5}}
$\boxed{8}$	{{4, 7, 9, 0}, {1, 4, 6, 9}, {7, 10, 0, 3}, {10, 1, 3, 6}}

Table 3.34. The SUM classes of set class 4-26.

T_n Transformations	**Isomorphism**	***Zn* Transformations**
$\{T_0, T_3, T_6, T_9\}$	⇔	Z0
$\{T_1, T_4, T_7, T_{10}\}$	⇔	Z4
$\{T_2, T_5, T_8, T_{11}\}$	⇔	Z8

Table 3.35. The homomorphism from the order-twelve T_n subgroup onto the order-three Z*n* subgroup for the tetrachords mediated by the isomorphism between the quotient group of the T_n group modulo $\{T_0, T_3, T_6, T_9\}$ and the order-three Z*n* subgroup for the tetrachords.

We also saw earlier with 3-4 that it is possible for non-symmetrical set classes to exhibit the same SUM-class structure as a symmetrical set class. In these problematic cases prime and inverted forms of the set class occupy the same SUM class, making it impossible to define any consistently operating SUM-class transformation within this single space. Of the tetrachords, set classes 4-4 and 4-18 exhibit this property for the $\{\boxed{0}, \boxed{4}, \boxed{8}\}$ SUM-class space and 4-13 and 4-14 for the $\{\boxed{2}, \boxed{6}, \boxed{10}\}$ SUM-class space. To create a generalized voice-leading space, we must segregate the prime and inverted forms into two separate spaces as we did with 3-4. Once this is done, we can then use the same groups we used for the non-symmetrical sets inhabiting $\{\boxed{1}, \boxed{3}, \boxed{5},$

$\boxed{7}$, $\boxed{9}$, $\boxed{11}$}. For the case of 4-18 seen in Tables 3.36 and 3.37, we set Δ^1 = L'P~L'R and Ω = L'RL'.

SUM Classes	Pitch-Class Set Members
$\boxed{0}$	{{0, 1, 4, 7}, {3, 4, 7, 10}, {6, 7, 10, 1}, {9, 10, 1, 4}}
$\boxed{4}$	{{1, 2, 5, 8}, {4, 5, 8, 11}, {7, 8, 11, 2}, {10, 11, 2, 5}}
$\boxed{8}$	{{2, 3, 6, 9}, {5, 6, 9, 0}, {8, 9, 0, 3}, {11, 0, 3, 6}}

Table 3.36. The SUM classes for the prime forms of set class 4-18.

SUM Classes	Pitch-Class Set Members
$\boxed{0'}$	{{5, 8, 11, 0}, {8, 11, 2, 3}, {11, 2, 5, 6}, {2, 5, 8, 9}}
$\boxed{4'}$	{{6, 9, 0, 1}, {9, 0, 3, 4}, {0, 3, 6, 7}, {3, 6, 9, 10}}
$\boxed{8'}$	{{7, 10, 1, 2}, {10, 1, 4, 5}, {1, 4, 7, 8}, {4, 7, 10, 11}}

Table 3.37. The SUM classes for the inverted forms of set class 4-18.

3.6 SUM Classes for Pentachords

The relationship between T_n, I_n, and the SUM-classes for five-note sets is defined as follows: $T_n(x \in \boxed{X}) = x \in \boxed{5n + X}$ and $I_n(x \in \boxed{X}) = x \in \boxed{5n - X}$. For the trichords and tetrachords we noted that several values of *n* would produce the same effect on the SUM classes because we are dealing with a mod-12 universe. This allowed us to collect several T_n and I_n transformations together into equivalence classes known as SUM-class transformations. The number of unique values of *zn* (where *z* is the cardinality of the set) is directly related to the relationship between *z* and 12. If *z* divides 12, then there will be exactly 12/*z* unique values of *zn*. If, however, *z* does not divide 12 (and is thus coprime with 12), then there will be 12 unique values of *zn*.[61] Such is the case with $z = 5$. What this means is that no two different values of *n* would cause T_n or I_n to send a given set to sets that belong to the same SUM class. Another way of thinking of this is that there are no order-five subgroups of the T_n/I_n group. As a result, all of the pentachords, whether symmetrical or not, will fill out the full SUM-class space of twelve SUM classes—symmetrical set classes containing one pitch-class set per SUM class and non-symmetrical sets containing one prime and one inverted set in each SUM class. What was exceptional for the non-symmetrical trichords and tetrachords (see 3-4, 4-4, 4-13, 4-14, and 4-18), then, is thus the norm (in fact, the only possibility) for non-symmetrical pentachords.

In the case of the symmetrical set classes like 5-35 (see Table 3.38), there will be no need to make use of the Y*n*/X*n* group, the contextual inversions or even the W*n* or I_n transformations since a subgroup of twelve Z*n* transformations alone will be sufficient to act simply transitively upon the space. The actions of this group on the SUM classes can be seen in Table 3.39. As before, this group is also directly related to the T_n group, but now by an isomorphism instead of homomorphism (see Table 3.40) since no two T_n transformations produce the same voice-leading interval when applied to a five-note set. Though this mapping is one-to-one, it does not necessarily map pairs of transformations with the same indexes. What this means is that to move to a pentatonic scale (for example) that lies a single semitone away in terms of a total voice-leading interval, we must transpose by T_5. This should remind us of the circle of fifths for the

[61] Alternatively, if $z > 6$ and is a multiple of an integer $x > 2$ that divides twelve, then there will be exactly 12/*x* values of *zn*.

diatonic scale in which the most "closely-related" scales are those that are related by T_5 or T_7. Indeed, we may create an essentially analogous structure for any five-note set by simply placing their SUM classes on a clock face. On these five-note clocks, though, clockwise motion around the circle produces a transposition by perfect fourth rather than by perfect fifth. We shall consider why this is so in the section on complementary set classes.

SUM Class	Pitch-Class Set Members
0	{10, 0, 2, 5, 7}
1	{3, 5, 7, 10, 0}
2	{8, 10, 0, 3, 5}
3	{1, 3, 5, 8, 10}
4	{6, 8, 10, 1, 3}
5	{11, 1, 3, 6, 8}
6	{4, 6, 8, 11, 1}
7	{9, 11, 1, 4, 6}
8	{2, 4, 6, 9, 11}
9	{7, 9, 11, 2, 4}
10	{0, 2, 4, 7, 9}
11	{5, 7, 9, 0, 2}

Table 3.38. The SUM classes of set class 5-35.

***Zn* SUM-class Transformations**	**Action on Sum Classes**
Z0	(0) (1) (2) (3) (4) (5) (6) (7) (8) (9) (10) (11)
Z1	(0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11)
Z2	(0, 2, 4, 6, 8, 10) (1, 3, 5, 7, 9, 11)
Z3	(0, 3, 6, 9) (1, 4, 7, 10) (2, 5, 8, 11)
Z4	(0, 4, 8) (1, 5, 9) (2, 6, 10) (3, 7, 11)
Z5	(0, 5, 10, 3, 8, 1, 6, 11, 4, 9, 2, 7)
Z6	(0, 6) (1, 7) (2, 8) (3, 9) (4, 10) (5, 11)
Z7	(0, 7, 2, 9, 4, 11, 6, 1, 8, 3, 10, 5)
Z8	(0, 8, 4) (1, 9, 5) (2, 10, 6) (3, 11, 7)
Z9	(0, 9, 6, 3) (1, 10, 7, 4) (2, 11, 8, 5)
Z10	(0, 10, 8, 6, 4, 2) (1, 11, 9, 7, 5, 3)
Z11	(0, 11, 10, 9, 8, 7, 6, 5, 4, 3, 2, 1)

Table 3.39. The actions of the order-twelve *Zn* subgroup on the twelve SUM classes.

***Zn* SUM-class Transformations**	**Isomorphism**	**Action on Sum Classes**
T_0	⇔	Z0
T_5	⇔	Z1
T_{10}	⇔	Z2
T_3	⇔	Z3
T_8	⇔	Z4
T_1	⇔	Z5
T_6	⇔	Z6
T_{11}	⇔	Z7
T_4	⇔	Z8
T_9	⇔	Z9
T_2	⇔	Z10
T_7	⇔	Z11

Table 3.40. The isomorphism from the order-twelve T_n subgroup to the order-twelve Z*n* subgroup.

All of the non-symmetrical pentachords will generate SUM-class spaces almost exactly like the one for 5-35 except that these SUM classes will each contain one prime form of the set as well as one inverted form of the set. In this way, all non-symmetrical pentachords generate SUM-class spaces that are analogous to the "exceptional" cases of 3-4, 4-4, 4-13, 4-14, and 4-18. As we saw in these cases, we will need to separate prime and inverted forms of the set into two separate spaces before we can define any transformations to act upon them. Once this is done, we can then use the full order twenty-four Y*n*/X*n* and Z*n*/W*n* transformations at the level of the

SUM class (with the inversional transformations moving between the two segregated spaces and the transpositions moving within each space) and the full $\Delta^n/\Delta^n\Omega$ and T_n/I_n groups at the level of the pitch-class set. Of course, to make use of the $\Delta^n/\Delta^n\Omega$ group we would also need to define the actions of P~, L~, and R~ on five-note sets. Such definitions become increasingly arbitrary and less generalizable as the size of the sets increase though, and so in general it will probably best to let the musical context dictate how these transformations should be defined rather than proposing definitions that will be meaningful for all pentachords.

3.7 SUM Classes for Hexachords

The T_n and I_n transformations act on the hexachords such that $T_n(x \in \boxed{X}) = x \in \boxed{6n + X}$ and $I_n(x \in \boxed{X}) = x \in \boxed{6n - X}$. Because 6 divides 12, there will be two unique values for 6*n*, both of which are congruent to 0 mod 6. Transpositionally-related sets will thus always belong to SUM classes that are congruent to one another mod 6 and inversionally-related sets will belong to SUM classes whose values sum to 0 mod 6. This gives us the following 6 possible variants of a hexachordal SUM-class space:

1) If SUM of the prime-form representative is congruent to 0 mod 6, all members of the set class will inhabit $\boxed{0}$ or $\boxed{6}$.

2) If SUM of the prime-form representative is congruent to 1 mod 6, all prime forms of the set class will inhabit $\boxed{1}$ or $\boxed{7}$ and all inverted forms will inhabit $\boxed{5}$ or $\boxed{11}$.

3) If SUM of the prime-form representative is congruent to 2 mod 6, all prime forms of the set class will inhabit $\boxed{2}$ or $\boxed{8}$ and all inverted forms will inhabit $\boxed{4}$ or $\boxed{10}$.

4) If SUM of the prime-form representative is congruent to 3 mod 6, all members of the set class will inhabit $\boxed{3}$ or $\boxed{9}$.

5) If SUM of the prime-form representative is congruent to 4 mod 6, all prime forms of the set class will inhabit $\boxed{4}$ or $\boxed{10}$ and all inverted forms will inhabit $\boxed{2}$ or $\boxed{8}$.

6) If SUM of the prime-form representative is congruent to 5 mod 6, all prime forms of the set class will inhabit $\boxed{5}$ or $\boxed{11}$ and all inverted forms will inhabit $\boxed{1}$ or $\boxed{7}$.

Cases 1 and 4 will be the only suitable abodes for the inversionally-symmetric hexachords, and in general we should expect the non-symmetrical set classes to take the form of cases 2, 3, 5 or 6. A complete inventory of the SUM-class structures for all six-note set classes can be seen Table 3.41, and, as we have seen before, this table reveals that not all set classes that generate a symmetrical space are necessarily symmetrical. In particular, we see that set classes 6-9, 6-14, 6-16, and 6-22 join the ranks of the other exceptional cases we have observed previously as non-symmetrical set classes that generate a "symmetrical" space. Unique to the hexachords, however, is the unusual case of 6-30 (the *Petrushka* chord) in which a pitch-class set with more than one degree of transpositional symmetry is not inversionally symmetrical. Even more interestingly, this set class is also the only example within the SUM-class universe of a symmetrical set class inhabiting an otherwise non-symmetrical SUM-class space. Its symmetric nature makes it so that the effects of several different transformations collapse onto one another, however, and so in the end the group structures that act on it will be the same as all other hexachords exhibiting the same SUM-class space. In other words, the SUM-class space still generalizes the voice-leading relationships within this set class even though it is quite unusual.

0, 6	3, 9	1, 5, 7, 11	2, 4, 8, 10
6-z4*	6-1*	6-z3	6-2
6-7*	6-z6*	6-5	6-z10
6-9	6-8*	6-z11	6-z12
6-16	6-z13*	6-18	6-15
6-z23*	6-14	6-z19	6-z17
6-z26*	6-20*	6-21	6-z24
6-z28*	6-22	6-z25	6-30
6-35*	6-z29*	6-27	6-31
6-z37*	6-32*	6-34	6-33
6-z45*	6-z38*	6-z36	6-z39
6-z48*	6-z42*	6-z40	6-z41
6-z49*	6-z50*	6-z44	6-z43
		6-z47	6-z46

Table 3.41. The hexachordal set classes that exhibit each of the four possible hexachordal SUM-class profiles. An asterisk (*) indicates that a set class is inversionally symmetrical.

Transformationally, these six-note systems will be analogous in every way to the systems for three- and four-note sets but with different transformational subgroups. All non-symmetrical set classes will make use of SUM-class-transformation subgroups of the form {TR0, TR6, IN0, IN6} where TR is either Y*n*, Z*n* and IN is either X*n*, W*n* while the symmetrical set classes will just use the {TR0, TR6} subgroups. The transformations at the level of the pitch-class set will be mapped to these SUM-class transformation subgroups via the cosets of the {T_0, T_2, T_4, T_6, T_8,

T_{10}} and {Δ^0, Δ^2, Δ^4, Δ^6, Δ^8, Δ^{10}} normal subgroups. As always, P~, L~, and R~ would need to be redefined to act on six-note sets, but at this point these transformations become so contextual as to hardly be worth defining at all.

3.8 SUM Classes for Complementary Set Classes

Thus far, we have examined all possible SUM-class structures for sets of cardinalities three through six. To understand the full extent of the SUM-class universe, we would of course want to know how these structures behaved in the context of larger sets as well. We may not necessarily have to investigate all of these cardinalities individually, however, in order to know what they will look like. If we know that a given set class is inversionally symmetrical or that it is z-related to another set, for example, then we also know that the "complement" of this set class will exhibit these same properties. If it could be proven that complementary set classes also generate the same SUM-class spaces, then, it would be possible to know exactly what a SUM-class space would like in any cardinality without actually having to go through the process of creating it.

The literal complement of any single pitch-class set is the set of pitch classes that are *not* in the given set. For example, the literal compliment of the fully-diminished seventh chord {0, 3, 6, 9} is the octatonic scale {1, 2, 4, 5, 7, 8, 10, 11}. One way to visualize this is with characteristic functions, which produce twelve-place binary vectors that simply indicate whether or not each of the twelve pitch classes is present in a particular set. The characteristic function of {0, 3, 6, 9} is thus (1, 0, 0, 1, 0, 0, 1, 0, 0, 1, 0, 0), which tells us that pitch classes 0, 3, 6, and 9 belong to this set and pitch classes 1, 2, 4, 5, 7, 8, 10, and 11 do not. This function also shows us that there are two empty places between each pair of pitch-classes in {0, 3, 6, 9}, and that all of these empty places together form an octatonic scale. If we understand that transposition by *n* has the effect of shifting all the values in the characteristic function *n* places to the right, then it should not be too difficult to see that transposing {0, 3, 6, 9} will not change the intervallic structure of its complement but merely the pitch classes it is built upon. For example, T_1 of {0, 3, 6, 9} is (0, 1, 0, 0, 1, 0, 0, 1, 0, 0, 1, 0). As we can see, there are still two empty places between each pair of pitch classes, and these empty places still form an octatonic scale when combined together: {0, 2, 3, 5, 6, 8, 9, 11}.

Inversion, likewise, does not actually change the intervals present in a pitch-class set but merely rearranges them. Here, for example, we take the characteristic functions of a D half-diminished seventh chord and a B major-minor seventh chord (which are inversions of each other about the B/C axis): (1, 0, 1, 0, 0, 1, 0, 0, 1, 0, 0, 0) and (0, 0, 0, 1, 0, 0, 1, 0, 0, 1, 0, 1). As can be seen, B major-minor seven has the same pattern of intervals (or gaps) as D half-diminished seven but just read backwards. This also means that the complements of these two sets will have the same intervallic structures as one another and thus belong to the same set class. Therefore, taking the complements of *any* two sets from the same set class will always produce two sets that also belong to the same set class. Or, put another way, all of the sets in one set class are the complements of all of the sets in another set class. In fact, the set classes are labeled in Allen Forte's system such that complementary set classes receive the same index number (except in the case of the hexachords).[62] The fully-diminished seventh chords and the octatonic scales, for example, are both index number 28 within their respective cardinalities.

[62] See Appendix 1 in Forte, *The Structure of Atonal Music*.

The hexachords are a special case because these are the only sets that can be their own complements. In such cases it is obvious that self-complementary hexachords will generate the same SUM-class space, but what about set classes that are not self-complementary? To investigate this, let us modify Table 3.41 so that there are arrows connecting all complementary set classes (see Table 3.42). If complementary set classes always produce the same SUM-class spaces, then we should not see any arrows connecting set classes in different columns, and, indeed, as Table 3.42 clearly reveals, this is the case.

0, 6	3, 9	1, 5, 7, 11	2, 4, 8, 10
6-z4*	6-1*	6-z3	6-2
6-7*	6-z6*	6-5	6-z10
6-9	6-8*	6-z11	6-z12
6-16	6-z13*	6-18	6-15
6-z23*	6-14	6-z19	6-z17
6-z26*	6-20*	6-21	6-z24
6-z28*	6-22	6-z25	6-30
6-35*	6-z29*	6-27	6-31
6-z37*	6-32*	6-34	6-33
6-z45*	6-z38*	6-z36	6-z39
6-z48*	6-z42*	6-z40	6-z41
6-z49*	6-z50*	6-z44	6-z43
		6-z47	6-z46

Table 3.42. The hexachordal set classes that exhibit each of the four possible hexachordal SUM-class profiles. An asterisk (*) indicates that a set class is inversionally symmetrical. Arrows connect complementary set classes, and all set classes without an arrow are self-complementary.

Intuitively, this is not particularly surprising given that complementary set classes are always similar in so many other ways (z-relations, inversional symmetry, etc.). But upon deeper reflection it is actually quite remarkable all of these sets of integers should sum to the *exact* same values as sets of their mod-12 complements. The reason this happens has to do with the SUM value of the chromatic aggregate. Summing the integers from 0 to 11 produces a value of six mod twelve, which means that any set and its literal complement will also sum to six mod twelve. In other words, the literal complement of any set in $\boxed{n}$ will be a set belonging to $\boxed{6 - n}$. Therefore, we can say that a set class and its complement will generate the same SUM-class space in the following case:

Theorem 3.1. Let *S* be the set of SUM classes generated by a single set class and *C* the set of SUM classes generated by the set-class complement of *S*. *S* and *C* will be equal if and only if for every $s \in S$, the SUM class $6 - s$ modulo 6 is also $\in S$.

Proof. It will be sufficient here to prove that the converse is true. If there were a SUM-class *s* in the SUM-class space *S* for which the inverse mod 6 was not also in *S*, then it is obvious that this SUM class would get sent to a SUM class in *C* that is not in *S* under the complement relation, which of course means that $S \neq C$.

Another way of seeing this is to think of the SUM classes as pitch classes and the SUM-class spaces as pitch-class sets. In these terms, if the set that represents the SUM-class space of a particular set class maps to itself under I_6, then the set class and its complement will generate the same SUM-class space. The hexachordal SUM-class spaces all map to themselves under I_6, and so we know that complementary spaces will always generate the same SUM-class space: $I_6(\{0, 6\}) = \{6, 0\}$; $I_6(\{3, 9\}) = \{3, 9\}$; $I_6(\{1, 5, 7, 11\}) = \{5, 1, 11, 7\}$; $I_6(\{2, 4, 8, 10\}) = \{4, 2, 10, 8\}$.

Similarly, we can see that all three-note set classes will generate the same SUM-class space as their nine-note complements because both of the possible trichord spaces are mapped to themselves under I_6: $I_6(\{0, 3, 6, 9\}) = \{6, 3, 0, 9\}$ and $I_6(\{1, 2, 4, 5, 7, 8, 10, 11\}) = \{5, 4, 2, 1, 11, 10, 8, 7\}$). The SUM-class spaces of complementary pentachords and heptachords will also be the same trivially because the pentachords always fill out all twelve SUM-classes. The non-symmetric tetrachords likewise generate the same SUM-class spaces as their eight-note complements since $I_6(\{1, 3, 5, 7, 9, 11\}) = \{5, 3, 1, 11, 9, 7\}$, but for the symmetrical tetrachords in SUM-class spaces $\boxed{0}$, $\boxed{4}$, $\boxed{8}$, and $\boxed{2}$, $\boxed{6}$, $\boxed{10}$, the complement relation actually sends complementary sets to the opposite space. Thus, the complement of a tetrachordal set class that generates the 0, 4, 8 space will inhabit the 2, 6, 10 space and vice versa. While it is true that these spaces occupy different SUM classes, the intervals *between* the SUM classes in both cases are the same, which means that the SUM-class transformations and voice-leading intervals between sets will still be the same for these set classes and their complements.

While the SUM-class transformations for a set and its complement will always be the same, this is not always be true of the transformations at the level of the pitch-class set. We already know that the relationship between voice-leading interval and the contextual transformations change from set class to set class, but it turns out that even the T_n transformations will produce different voice-leading intervals within complementary set classes. Consider, for example, the SUM-class spaces of 5-35 (the usual pentatonic scale) and 7-35 (the diatonic scale) seen in Table 3.43. Reading down both columns in the table produces a chain of T_5 transformations, but, as can be seen, a T_5 chain produces *ascending* voice leading for the pentatonic scales but produces descending voice leading for the diatonic scales.

	5-35	Complement	7-35	
	$\boxed{0}$ = {10, 0, 2, 5, 7}	⇔	$\boxed{6}$ = {3, 4, 6, 8, 9, 11, 1}	
T_5	$\boxed{1}$ = {3, 5, 7, 10, 0}	⇔	$\boxed{5}$ = {8, 9, 11, 1, 2, 4, 6}	T_5
T_5	$\boxed{2}$ = {8, 10, 0, 3, 5}	⇔	$\boxed{4}$ = {1, 2, 4, 6, 7, 9, 11}	T_5
etc.	$\boxed{3}$ = {1, 3, 5, 8, 10}	⇔	$\boxed{3}$ = {6, 7, 9, 11, 0, 2, 4}	etc.
	$\boxed{4}$ = {6, 8, 10, 1, 3}	⇔	$\boxed{2}$ = {11, 0, 2, 4, 5, 7, 9}	
	$\boxed{5}$ = {11, 1, 3, 6, 8}	⇔	$\boxed{1}$ = {4, 5, 7, 9, 10, 0, 2}	
	$\boxed{6}$ = {4, 6, 8, 11, 1}	⇔	$\boxed{0}$ = {9, 10, 0, 2, 3, 5, 7}	
	$\boxed{7}$ = {9, 11, 1, 4, 6}	⇔	$\boxed{11}$ = {2, 3, 5, 7, 8, 10, 0}	
	$\boxed{8}$ = {2, 4, 6, 9, 11}	⇔	$\boxed{10}$ = {7, 8, 10, 0, 1, 3, 5}	
	$\boxed{9}$ = {7, 9, 11, 2, 4}	⇔	$\boxed{9}$ = {0, 1, 3, 5, 6, 8, 10}	
	$\boxed{10}$ = {0, 2, 4, 7, 9}	⇔	$\boxed{8}$ = {5, 6, 8, 10, 11, 1, 3}	
	$\boxed{11}$ = {5, 7, 9, 0, 2}	⇔	$\boxed{7}$ = {10, 11, 1, 3, 4, 6, 8}	

Table 3.43. The SUM classes of set classes 5-35 and 7-35 arranged so that sets in the same row are complements of one another.

This strange reversal has to do with the way that complementary sets interact with the SUM-class system. We already know that the SUM-class spaces for these two set classes *as wholes* map onto one another under the complement relationship, but this does not mean that any particular SUM class in 5-35 necessarily maps to the same SUM class in 7-35. Indeed, we noted earlier that complementary sets belong to set classes that are each other's inverses *mod 6*, which means that only SUM classes $\boxed{3}$ and $\boxed{9}$ will map to themselves. The rest of the SUM classes will then reflect to their mirror image around these axes of symmetry. For example, [2], which right *below* [3], is the mod-six complement of [4], which is right *above* [3]. The result of all this is that any pair of pitch-class sets in 5-35 that lie *n* semitones apart (in terms of a PVLS) are the complements of a pair of pitch-class sets in 7-35 that lie $12 - n$ semitones apart. In other words, to proceed in the same direction within these two spaces would require the use of *complementary* transformations. Those sets that lie six semitones apart in one set class will be the complements of sets that also lie six semitones apart in the other set class because six is its own complement mod 12 (see, for example, the interval between {10, 0, 2, 5, 7} and {4, 6, 8, 11, 1} and between {3, 4, 6, 8, 9, 11, 1} and {9, 10, 0, 2, 3, 5, 7}). Because all same-quality hexachords *always* lie six semitones apart, then, the T_n transformations will always produce the same voice-leading intervals in any hexachordal set class and its complement. For all other cardinalities, however, the same T_n transformation will produce complementary voice-leading intervals in any set class and its complement.

Aside from this one small difference, the SUM-class space of any seven-, eight-, or nine-note set class is essentially the same as that of its complement, and what we have discussed for the trichords, tetrachords, and pentachords, is thus also true of the heptachords, octachords, and nonachords. The only real obstacle in the way of a functional SUM-class system for any of these larger set classes, then, is an analytically meaningful definition of the contextual inversions. But there is also likely to be a difference in the role that these larger sets would play in a musical context. With the smaller cardinalities we were interested the voice-leading intervals in terms of actual motion from one *sonority* to another, but examples of surface-level progressions between very large sonorities are significantly harder to come by (though they certainly do exist). Where we will be most likely to encounter these larger sets, then, is in the context of a collection or scale rather than as a sonority. In this context, a SUM-class system would measure how "closely-related" two collections are, much like we speak of the "relatedness" of various major and minor keys. In fact, we often discuss these key relations in terms of distances on the circle of fifths, and *the circle of fifths is identical to the SUM-class system for 7-35* but without the notion of the SUM classes.[63] The SUM-class systems for some of these larger set classes could be used to create analogous structures for non-tonal contexts.

3.9 Summary

This chapter has explored all possible forms that a SUM-class space can take when built from the pitch-class sets of a single set class. In order to determine the type of SUM-class space that a set class will generate, we must only know the kinds of mappings that the T_n and I_n transformations will produce secondarily at the level of the SUM classes. In general, if z is the cardinality of the set, then we can say that $T_n(x \in \boxed{X}) = x \in \boxed{zn + X}$ and $I_n(x \in \boxed{X}) = x \in \boxed{zn - X}$.

For values of z that divide 12, there will only be $12/z$ unique values for zn and thus $12/z$ sets of z transformations whose actions produce the same movement within the SUM-class space. These will be the cosets that are isomorphic to the Z*n* and W*n* SUM-class transformations. A set of contextual inversions can also be defined on each set class that make it possible to generalize voice-leading intervals between inversionally-related sets. These transformations will congregate in sets of z into $12/z$ sets that produce the same motion within the SUM-class space. These sets are the cosets that are isomorphic to the Y*n* and X*n* SUM-class transformations. Pitch-

[63] The notion of SUM class is rather trivial for the symmetrical pentachords and heptachords anyway since there is only ever one pitch-class set in each SUM class.

class sets related by transposition will always belong to SUM classes that are congruent to one another mod *z* and sets related by inversion will always belong to SUM classes that sum to 0 mod *z*. Inversionally-symmetrical set classes (whose sets are related to one another by transposition *and* inversion) can thus only inhabit those SUM classes that are congruent to 0 mod *z* or that sum with themselves to produce 0 mod *z*. If *any* set in the set class sums to 0 mod *z* or to a value that sums with itself to produce 0 mod *z* (i.e., a number that is its own inverse mod *z*), then all sets in the set class will belong to SUM classes congruent to this value mod *z*. In such cases, the set class will only generate 12/*z* SUM classes and will either be inversionally-symmetric or an exceptional case where prime and inverted forms of a non-symmetric set class are placed into the same SUM classes. If there is *any* set in the set class that does not sum to 0 mod *z* or is not its own inverse mod *z*, then the set class is not inversionally symmetrical and there will be 12/*z* SUM classes that contain the prime forms of the set class and 12/*z* SUM classes that contain the inverted forms of the set class.

For values of *z* that do not divide 12 but are multiples of an integer $x > 2$ that *does* divide 12, there will be 12/*x* unique values of *zn* and 12/*x* sets of *x* transformations whose actions produce the same movement within the SUM-class space. That is, set classes with cardinalities greater than six will generate SUM-class spaces that are the same as the SUM-class spaces of their complements. Finally, for values of *z* that are coprime with 12, there will be no values of *zn* that will be the same, in which case the members of the set class will inhabit the full SUM-class space and all non-symmetrical set classes will place prime and inverted forms of the set class into the same SUM classes.

This chapter has been concerned only with sets of cardinalities three through nine, but as the above generalizations show, it is also possible to create SUM-class spaces for sets of *any* cardinality. In fact, the twelve pitch classes are essentially a SUM-class system of their own on all one-note sets and the T_n and I_n transformations are equivalent to SUM-class transformations on this space. By our complement rule, we also know that the eleven note sets will inhabit this same SUM-class space—which is the same as the spaces for five- and seven-note sets. The dyads and their ten-note complements similarly inhabit spaces that are either the same or have the same intervallic structure as the SUM-class spaces of the non-symmetrical tetrachords. We have seen throughout this thesis that two set classes that produce the same SUM-class spaces possess the same internal voice-leading intervals between their members. What is implied by the sameness of these various SUM-class spaces, then, is that voice leading is somehow the same within each of these contexts; that there is something analogous or congruent about " . . . a characteristic directed measurement, distance, or motion . . . "[64] in one set class and a set class with the same SUM-class structure. We might say, for example, that there is something "equivalent" or at least "congruent" about the motion from G+ to C+ and from {1, 2, 5} to {2, 3, 6} even though these events take place within totally different set classes. This suggest, perhaps, that SUM-class systems need not be confined to a single set class but might instead be constructed for small sets of closely-related set classes or even entire cardinalities. The remainder of this thesis shall consider such possibilities.

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[64] Lewin, *Generalized Musical Intervals,* xxix.

Chapter 4: SUM-Class Systems for Whole Cardinalities

Chapter 3 explored the SUM-class system outside the context of the major and minor triads. While these systems were different from the system for the consonant triads in many superficial ways (inhabiting different SUM classes, requiring different transformational groups), the overall relationship between SUM class and voice-leading intervals nevertheless remained the same. What we found was that the total voice-leading interval between any two sets was always equal to the difference between the SUM classes they belonged to, regardless of the set class this system was defined upon. And this relationship between SUM class and voice-leading interval continued to be true even in cases like 3-8 where transformations at the level of the pitch-class set were unable to capture it. In other words, the fact that we were able to define a transformation to take us from {9, 1, 3} to {9, 11, 3} but not from {9, 1, 3} to {10, 0, 1} does not mean that we cannot say anything about the voice-leading interval from {9, 1, 3} to {10, 0, 1}.

This suggest, perhaps, that our insistence on having well-defined transformations at every level of organization has actually been serving to obscure what is otherwise a very clear relationship between SUM class and voice leading, and as a corollary that we need not limit the SUM-class system to single set classes. Clearly, the distinction between SUM class and pitch-class set and between SUM-class transformation and pitch-class set transformation is a fundamental one, and in no way am I advocating for any slackening in the rigor with which we have proceeded thus far. What I am suggesting, however, is that we might be less concerned with many of the surface-level details if we are primarily concerned with voice-leading generalizations.

To begin, let us once again consider the SUM-class space for the consonant triads, which is reproduced below as Table 4.1. As the SUM classes of this space reveal (since they have the same values as the pitch classes of an octatonic scale), adjacent SUM classes within this space may be separated by either a semitone or a whole tone. The voice leading from {0, 4, 7} to {0, 3, ,7} (which inhabit adjacent SUM classes), for example, requires only that one voice move by semitone, whereas the voice leading from {0, 4, 7} to {1, 4, 8} (which also inhabit adjacent SUM classes) requires two voices to move by semitone. The pitch-class sets inhabiting SUM classes that lie a single semitone apart are thus as "close" as possible to one another in terms of a voice-leading interval without belonging to the same SUM class. But between the sets that inhabit SUM classes separated by a whole tone there is a "gap" of sorts, which suggests that there might exist a pitch-class set or even several pitch-class sets that would lie one semitone from the sets on either side and thus help to "fill in" this gap. That is, there should be some set that would be a semitone away from {0, 4, 7} *and* from {1, 4, 8}. Clearly, however, there is no major or minor triad that could play this role, and so we will have to look outside the 3-11 set class for these mystery sets.

SUM Class	**Pitch-Class Sets**
1	{{1, 4, 8}, {5, 8, 0}, {9, 0, 4}}
2	{{5, 9, 0}, {9, 1, 4}, {1, 5, 8}}
4	{{2, 5, 9}, {6, 9, 1}, {10, 1, 5}}
5	{{6, 10, 1}, {10, 2, 5}, {2, 6, 9}}
7	{{3, 6, 10}, {7, 10, 2}, {11, 2, 6}}
8	{{7, 11, 2}, {11, 3, 6}, {3, 7, 10}}
10	{{0, 3, 7}, {4, 7, 11}, {8, 11, 3}}
11	{{8, 0, 3}, {0, 4, 7}, {4, 8, 11}}

Table 4.1. The SUM-class space for the consonant triads, set class 3-11.

Thanks to Cohn, however, we need not look far, for he notes that the sets on either side of these whole-tone gaps are exactly those sets that belong to the four "Weitzmann" regions that can be generated by a single semitonal perturbation of an augmented triad.[65] Indeed, the {0, 4, 8} augmented triad does lie exactly one semitone away from {0, 4, 7} and from {1, 4, 8} (as well as from the two other triads on either side), and thus bridges the gap between them. Furthermore, as we have already seen, the four augmented triads sum to *exactly* the four values that are missing from the SUM-class system for the consonant triads (0, 3, 6, and 9). Cohn thus suggests that the SUM-class system for the consonant triads could be "augmented" (to use his pun) to include the

[65] Cohn, "Square Dances," 293–94. For more in Weitzmann regions see Richard Cohn, "Weitzmann's Regions, My Cycles, and Douthett's Dancing Cubes," *Music Theory Spectrum* 22, no. 1 (2000): 89–103.

augmented triads and make it possible to navigate through the entire space by semitone.[66] This new compound space can be seen in Table 4.2.

SUM Class	**Pitch-Class Sets**
0	{0, 4, 8}
1	{{1, 4, 8}, {5, 8, 0}, {9, 0, 4}}
2	{{5, 9, 0}, {9, 1, 4}, {1, 5, 8}}
3	{1, 5, 9}
4	{{2, 5, 9}, {6, 9, 1}, {10, 1, 5}}
5	{{6, 10, 1}, {10, 2, 5}, {2, 6, 9}}
6	{2, 6, 10}
7	{{3, 6, 10}, {7, 10, 2}, {11, 2, 6}}
8	{{7, 11, 2}, {11, 3, 6}, {3, 7, 10}}
9	{3, 7, 11}
10	{{0, 3, 7}, {4, 7, 11}, {8, 11, 3}}
11	{{8, 0, 3}, {0, 4, 7}, {4, 8, 11}}

Table 4.2. The compound SUM-class space for the consonant and augmented triads.

[66] Cohn, "Square Dances," 294.

Cohn then suggests a group of twelve transformations to move through the SUM classes of this space that are identical to the subgroup of twelve Z*n* transformations we used for the pentachords and heptachords.[67] Just as in that context, these twelve transformations act simply transitively upon the twelve SUM classes and can thus define a GIS. But we must be careful to remember that this does not mean that we have create a GIS for the consonant and augmented triads—only for the twelve SUM classes. The Z*n* transformations are only defined to act on SUM classes, and so it would be wrong to say that {0, 4, 7} and {0, 4, 8} (or any other pitch-class sets from $\boxed{11}$ and $\boxed{0}$) are related to one another by any Z*n* or that any Z*n* transforms {0, 4, 7} into {0, 4, 8}. Instead, what the GIS tell us is that there is an element $\boxed{0}$ in the SUM-class space set that lies the interval of Z1 from the element $\boxed{1}$ in the same set, but it says nothing about what that interval might mean for the pitch-class sets contained within the SUM classes. Only by explicitly linking SUM classes to PVLS values can these intervals be related to voice-leading intervals between pitch-class sets. But of course, Cohn already did this for us when he proved that the PVLS between two pitch-class sets was simply the difference in their SUM values.[68] Thus, we can say that the voice-leading interval between any pair of sets (*a*, *b*) from Table 4.2 will be equal to *n* of the Z*n* transformation that moves between the SUM classes these sets inhabit. That is, the interval from *a* to *b* = $Z_{\text{PVLS(SUM}(a), \text{ SUM}(b))} = Z_{(\text{SUM}(b) - \text{SUM}(a))}$.

But because the consonant and augmented triads belong to different set classes, there are no T_n or I_n transformations, contextual or otherwise, that will be able to map between these set

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[67] Cohn labels these transformations as T_n, but since we have also used the T_n pitch-class transformations at the level of the pitch-class set, I have elected to continue to refer to these SUM class transformations as the Z*n* transformations. See Cohn, "Square Dances," 294.

[68] Cohn, "Square Dances," 286.

classes at the level of the pitch-class set.[69] That is, while it is possible to say what the total voice-leading distance between any two sets within this space will be, it is *not* possible to generalize the kinds of transformations (or motions) that will have to take place at the level of the individual voice in order to achieve this total distance. In a compound SUM-class space like this one, then, we will have to be content to transformationally define voice-leading relationships only at the level of the SUM classes.

This need not necessarily a bad thing though. As we discussed in the introduction to this chapter, defining transformations at the level of the pitch-class set has often required a number of additional levels of abstraction, and in the end these transformations have not usually provided us with any meaningful information beyond what was already revealed by the SUM-class transformations. In fact, our main goal has usually been to show exactly the ways in which distinct pitch-class set transformations were actually identical to one another from the perspective of voice leading. Furthermore, it is evident that the compound space for the consonant and augmented triads is capturing important information about voice leading between these two set classes, and so to discard this space simply because we are unable to define a satisfactory set of transformations at the surface level seems rather foolhardy. In short, a lack of

[69] I experimented for a time with a set of contextually-defined transpositions that only affect a single pitch class at a specific order position within each set. See Brandon Derfler, "Single-Voice Transformations: A Model for Parsimonious Voice Leading" (PhD diss., University of Washington, 2007). Transposing the third order position of any set up by semitone, for example, creates a chain of alternating major, minor, and augmented triads that moves through all twelve SUM classes. In fact, I was able to create eight such transformations that made it possible to move from *one* set in one SUM class to *one* set in any of the other eleven SUM classes. Unfortunately, however, these transformations did not allow for motion between one set and *any* other set and were thus unable to create a simply-transitive group of pitch-class set transformations that could have been contained within the SUM-class transformations.

pitch-class set transformations does not necessarily limit our analytical power, but merely forces us to make our observations at a higher level of abstraction.

If we concede that a compound SUM-class system can be built from two set classes that are not transformationally related, then we might wonder whether other set classes might also be able to fill in the gaps of 3-11's SUM-class system. We saw in Chapter 3 that all symmetrical trichords (and even 3-4) generate the same SUM-class spaces, which means that any of these set classes will also occupy the SUM classes left empty by 3-11. But does this also mean that the sets in these SUM classes will lie the same distance from the major and minor triads as the augmented triads did? In many ways, actually, the augmented triads are not particularly representative of the symmetrical trichords in general. Indeed, as *the* maximally-even trichord, the augmented triads hold a very special place within the trichordal universe—especially in their relationship to the consonant triads—and so it probably should not surprise us that these sets can fill these whole-tone gaps so easily.[70] But is it just a coincidence that these sets also inhabit the SUM classes left empty by the consonant triads, or is the SUM-class system actually capturing an important relationship between voice leading and SUM class that is irrespective of set class?

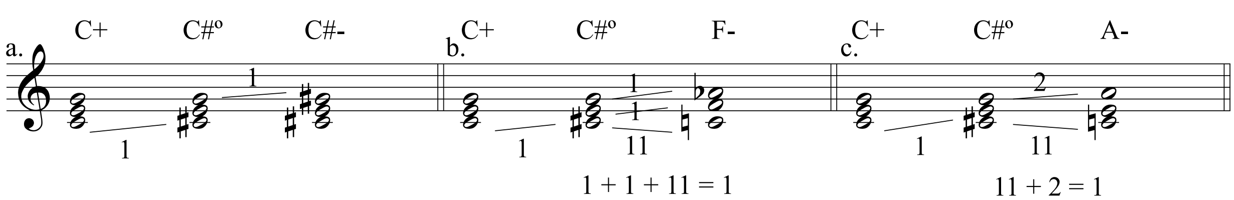
To investigate this, Table 4.3 combines the SUM-class spaces of the consonant and diminished triads. The diminished triads also inhabit $\boxed{0}$, $\boxed{3}$, $\boxed{6}$, and $\boxed{9}$, but here there are three diminished triads in each SUM class instead of only one. Each of these diminished triads can be produce by a semitonal perturbation of *one* of the six consonant triads surrounding it, but not from *all* six as we saw with the augmented triad. It is not immediately clear, then, that any one

[70] For more on "maximally-even sets," see John Clough and Jack Douthett, "Maximally Even Sets," *Journal of Music Theory* 35, no. 1/2 (1991): 93–173.

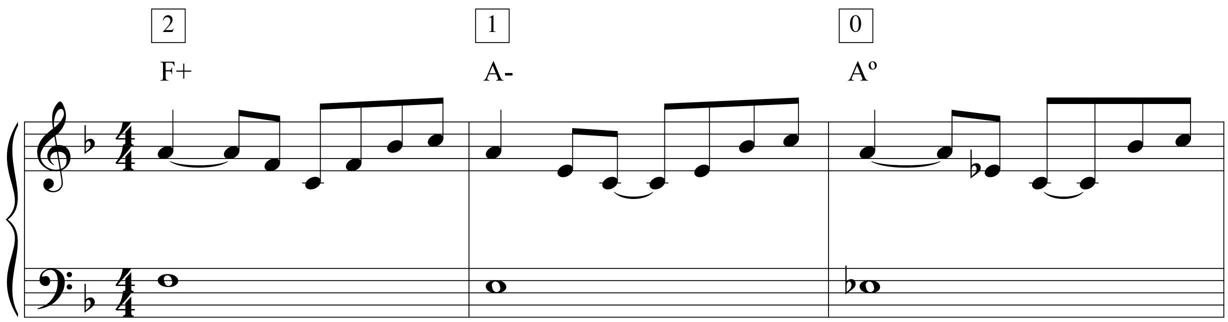
diminished triad would necessarily lie the same distance from all six consonant triads on either side of it. And yet, as Example 4.1 reveals, {1, 4, 7} is able to fill in the gap between {0, 4, 7} and all three minor triads in class 1 in such a way that the PVLS between adjacent triads is always 1 ({9, 0, 3} and {5, 8, 11} would function in much the same way)! A quick test will show that the interval from any diminished triad to any three major or minor triads in the same SUM class will *always* be the same. The SUM-class space made from the union of the consonant and diminished triads, then, is essentially equivalent to the space made from the union of the consonant and augmented triads from the perspective of voice leading. Examples of augmented and diminished triads as semitonal perturbations of major and minor chords are actually quite common in popular music. See, for example, the two chord progressions from David Bowie's *Life on Mars?* shown in Examples 4.2 and 4.3.

SUM Class	Pitch-Class Sets
0	{{1, 4, 7}, {9, 0, 3}, {5, 8, 11}}
1	{{1, 4, 8}, {5, 8, 0}, {9, 0, 4}}
2	{{5, 9, 0}, {9, 1, 4}, {1, 5, 8}}
3	{{6, 9, 0}, {2, 5, 8}, {10, 1, 4}}
4	{{2, 5, 9}, {6, 9, 1}, {10, 1, 5}}
5	{{6, 10, 1}, {10, 2, 5}, {2, 6, 9}}
6	{{7, 10, 1}, {3, 6, 9}, {11, 2, 5}}
7	{{3, 6, 10}, {7, 10, 2}, {11, 2, 6}}
8	{{7, 11, 2}, {11, 3, 6}, {3, 7, 10}}
9	{{0, 3, 6}, {8, 11, 2}, {4, 7, 10}}
10	{{0, 3, 7}, {4, 7, 11}, {8, 11, 3}}
11	{{8, 0, 3}, {0, 4, 7}, {4, 8, 11}}

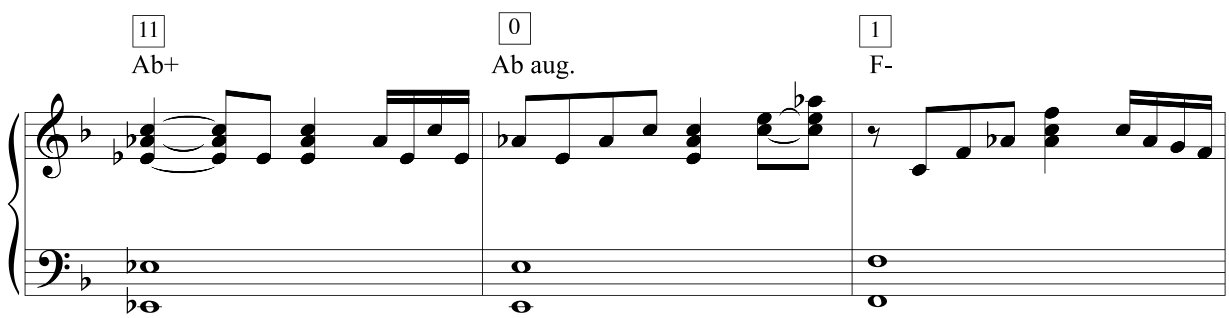
Table 4.3. The compound SUM-class space for the consonant and diminished triads.



Example 4.1. C#º filling in the “gap” from C+ to C#- (a), F- (b), and A- (c).

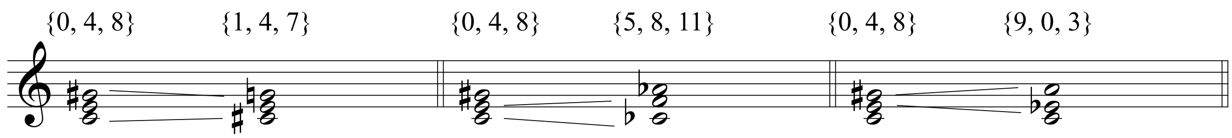


Example 4.2. Semitonal voice leading in David Bowie, *Life on Mars?*, mm. 1–3.



Example 4.3. Semitonal voice leading in David Bowie, *Life on Mars?*, mm. 17–19.

Since both the augmented and diminished triads can be combined with the consonant triads to create a compound system, we might also wonder if it also be possible to combine their spaces as well. We already know that these two set classes inhabit the same SUM classes, and we also know that voice leading between sets in the same SUM class will always involve same-interval contrary motion in two voices when the sets in question belong to the same set class. As can be seen in Example 4.4, this same kind of voice leading occurs between any augmented and diminished triads that inhabit the same SUM classes, and so a compound space for these two set classes will be equivalent to the spaces for either set class on its own. Even more importantly, though, this augmented/diminished compound space can also be united with the consonant triads to create an even larger compound SUM-class space that generalizes the voice leading between all four triad types.



Example 4.4. Contrary motion between the augmented and diminished triads inhabiting $\boxed{0}$.

The diminished and augmented triads are certainly the most likely candidates for “gap fillers” between consonant triads, but the SUM-class system is so powerful in its ability to generalize voice-leading intervals that we could actually fill the gaps in 3-11’s SUM-class system with any set class that inhabits $\boxed{0}$, $\boxed{3}$, $\boxed{6}$, and $\boxed{9}$. What this suggest, is that set-class membership really is not germane to voice-leading intervals when conceived of as a PVLS. Thus, whether two particular SUM classes each contain one set or one-hundred sets (or even if they contain different numbers of sets), and whether these sets are all from the same set class or all from different set classes, we can know that *all* of these sets will lie the same voice-leading interval from one another.

There is no reason, therefore, to continue to limit SUM-class spaces to the sets of a single set class, and so we formally construct a SUM-class system for all three-note sets via the relation in Definition 4.1.

Definition 4.1. Let *S* be the set of all possible pitch-class sets of three different mod-12 integers and *R* a relation on *S* such that $(a, b) \in R$ for any $a, b \in S$ that satisfy the equation $\text{SUM}(a) = \text{SUM}(b)$.

This definition is obviously very similar to the original definition of a SUM-class space in Chapter 1, but now without reference to set class, which allows it to be sufficiently general for an entire cardinality. Since this relation invokes the usual notion of equivalence, it is easy to see that it is an equivalence relation and as such partitions the universe of three-note pitch-class sets (which is to say that every possible three-note pitch-class set will belong to one and only one SUM class). A complete inventory of this "super" SUM-class system for cardinality three can be seen in Table 4.4.

SUM Class	**Pitch-Class Sets**
0	{{11, 0, 1}, {3, 4, 5}, {7, 8, 9}, {2, 3, 7}, {9, 1, 2}, {6, 7, 11}, {1, 5, 6}, {10, 11, 3}, {5, 9, 10}, {2, 4, 6}, {10, 0, 2}, {6, 8, 10}, {1, 3, 8}, {5, 7, 0}, {9, 11, 4}, {1, 4, 7}, {9, 0, 3}, {5, 8, 11}, {0, 4, 8}}
1	{{3, 4, 6}, {7, 8, 10}, {11, 0, 2}, {10, 1, 2}, {2, 5, 6}, {6, 9, 10}, {2, 3, 8}, {6, 7, 0}, {10, 11, 4}, {2, 4, 7}, {6, 8, 11}, {10, 0, 3}, {9, 1, 3}, {1, 5, 7}, {5, 9, 11}, {1, 4, 8}, {5, 8, 0}, {9, 0, 4}}
2	{{11, 1, 2}, {3, 5, 6}, {7, 9, 10}, {3, 4, 7}, {7, 8, 11}, {11, 0, 3}, {9, 2, 3}, {1, 6, 7}, {5, 10, 11}, {10, 1, 3}, {2, 5, 7}, {6, 9, 11}, {2, 4, 8}, {6, 8, 0}, {10, 0, 4}, {5, 9, 0}, {9, 1, 4}, {1, 5, 8}}
3	{{0, 1, 2}, {4, 5, 6}, {8, 9, 10}, {3, 4, 8}, {10, 2, 3}, {7, 8, 0}, {2, 6, 7}, {11, 0, 4}, {6, 10, 11}, {3, 5, 7}, {11, 1, 3}, {7, 9, 11}, {5, 10, 0}, {2, 4, 9}, {6, 8, 1}, {6, 9, 0}, {2, 5, 8}, {10, 1, 4}, {1, 5, 9}}
4	{{0, 1, 3}, {4, 5, 7}, {8, 9, 11}, {11, 2, 3}, {3, 6, 7}, {7, 10, 11}, {3, 4, 9}, {7, 8, 1}, {11, 0, 5}, {3, 5, 8}, {7, 9, 0}, {11, 1, 4}, {6, 10, 0}, {10, 2, 4}, {2, 6, 8}, {2, 5, 9}, {6, 9, 1}, {10, 1, 5}}

continued

Table 4.4. The "super" SUM-class space for all three-note sets.

Table 4.4 continued

SUM Class	Pitch-Class Sets
5	{{0, 2, 3}, {4, 6, 7}, {8, 10, 11}, {0, 1, 4}, {4, 5, 8}, {8, 9, 0}, {6, 11, 0}, {10, 3, 4}, {2, 7, 8}, {7, 10, 0}, {11, 2, 4}, {3, 6, 8}, {3, 5, 9}, {7, 9, 1}, {11, 1, 5}, {6, 10, 1}, {10, 2, 5}, {2, 6, 9}}
6	{{1, 2, 3}, {5, 6, 7}, {9, 10, 11}, {0, 1, 5}, {7, 11, 0}, {4, 5, 9}, {11, 3, 4}, {8, 9, 1}, {3, 7, 8}, {0, 2, 4}, {8, 10, 0}, {4, 6, 8}, {6, 11, 1}, {3, 5, 10}, {7, 9, 2}, {7, 10, 1}, {3, 6, 9}, {11, 2, 5}, {2, 6, 10}}
7	{{1, 2, 4}, {5, 6, 8}, {9, 10, 0}, {8, 11, 0}, {0, 3, 4}, {4, 7, 8}, {0, 1, 6}, {4, 5, 10}, {8, 9, 2}, {0, 2, 5}, {4, 6, 9}, {8, 10, 1}, {7, 11, 1}, {11, 3, 5}, {3, 7, 9}, {3, 6, 10}, {7, 10, 2}, {11, 2, 6}}
8	{{9, 11, 0}, {1, 3, 4}, {5, 7, 8}, {1, 2, 5}, {5, 6, 9}, {9, 10, 1}, {7, 0, 1}, {11, 4, 5}, {3, 8, 9}, {8, 11, 1}, {0, 3, 5}, {4, 7, 9}, {0, 2, 6}, {4, 6, 10}, {8, 10, 2}, {7, 11, 2}, {11, 3, 6}, {3, 7, 10}}
9	{{10, 11, 0}, {2, 3, 4}, {6, 7, 8}, {1, 2, 6}, {8, 0, 1}, {5, 6, 10}, {0, 4, 5}, {9, 10, 2}, {4, 8, 9}, {1, 3, 5}, {9, 11, 1}, {5, 7, 9}, {0, 2, 7}, {4, 6, 11}, {8, 10, 3}, {0, 3, 6}, {8, 11, 2}, {4, 7, 10}, {3, 7, 11}}
10	{{2, 3, 5}, {6, 7, 9}, {10, 11, 1}, {9, 0, 1}, {1, 4, 5}, {5, 8, 9}, {1, 2, 7}, {5, 6, 11}, {9, 10, 3}, {1, 3, 6}, {5, 7, 10}, {9, 11, 2}, {8, 0, 2}, {0, 4, 6}, {4, 8, 10}, {0, 3, 7}, {4, 7, 11}, {8, 11, 3}}
11	{{10, 0, 1}, {2, 4, 5}, {6, 8, 9}, {2, 3, 6}, {6, 7, 10}, {10, 11, 2}, {8, 1, 2}, {0, 5, 6}, {4, 9, 10}, {9, 0, 2}, {1, 4, 6}, {5, 8, 10}, {1, 3, 7}, {5, 7, 11}, {9, 11, 3}, {8, 0, 3}, {0, 4, 7}, {4, 8, 11}}

Note that 0, 3, 6, and 9 each contain nineteen pitch-class sets whereas the rest of the SUM classes only contain eighteen sets. It is actually rather surprising these totals are so similar given that 0, 3, 6, and 9 are the abodes of the symmetrical set classes, which only generate half as many unique sets as the non-symmetrical sets. Furthermore, there are also seven non-symmetrical trichordal set classes and only five symmetrical set classes. The only reason this super SUM class spaces is evenly distributed at all, then, is because of the exceptional case of 3-4, which places all twenty-four of its unique forms into $\boxed{0}$, $\boxed{3}$, $\boxed{6}$, and $\boxed{9}$. Thus, one could almost imagine that there “had” to be some sort of hybrid between the symmetrical and non-symmetrical trichords in order for the entire three-note universe to be balanced.

We could also create super SUM-class spaces for any other cardinality by simply modifying the reference to cardinality in Definition 4.1. The spaces for any of the larger cardinalities are so expansive as to be impractical to display in the manner of Table 4.4, but it is not particularly important to have a list of every set in each SUM class anyway since we know that these systems are all identical at the level of the SUM class and that finding voice-leading intervals between any two pitch-class sets is simply a matter of taking the difference between their SUMs.

Of all the super SUM-class systems, though, the most suggestive is the system for the one-note sets seen in Figure 4.1. This space ought to look quite familiar to us because it is *exactly* the same as pitch-class space itself. This not only shows us that we area already quite familiar with many aspects of these super SUM-class systems, but also implies that super SUM-class systems are somehow analogous to the pitch-class universe (since they all have the same SUM-class structure as the system for one-note sets). In fact, I believe that the analogy between the pitch-class and super SUM-class universes has much to teach us about the SUM-class system and why it is worth invoking.

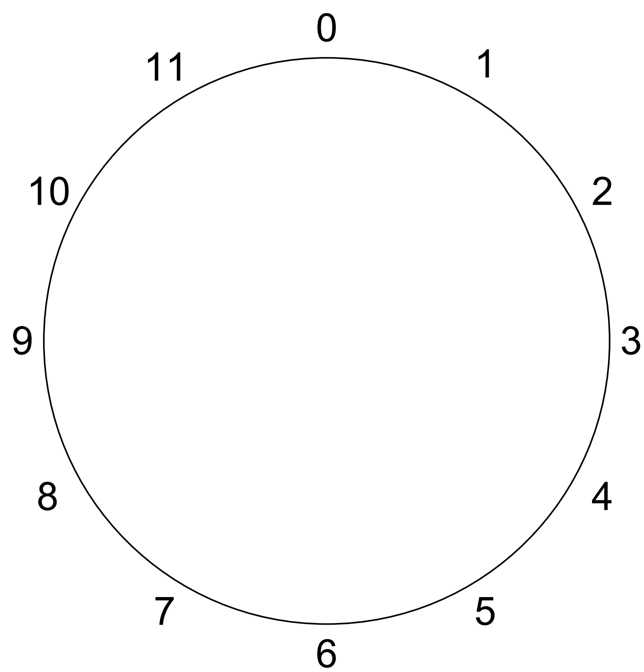


Figure 4.1. The SUM-class space for all one-note sets mapped onto a clock face.

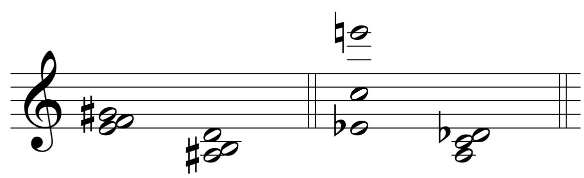
Let us begin by once again discussing the many assumptions at the heart of the concept of a pitch class. The most fundamental of these assumptions is that humans perceive all power-of-two multiples of the same frequency as the same fundamental "pitch." This suggest, for example, that a melody sung by a man and a boy the within their respective registers is will be perceived as the same melody, which is easy enough for us to accept, but true octave equivalence suggests that the two Cs at the extreme ends of the piano are also congruent to one another. Though I imagine that the vast majority of people would be unable to hear these two pitches as "the same," we are willing to consider them to be equivalent to one another because doing so allows us to compress the infinite pitch universe into something that is conceptually manageable. While it is true that we lose some specificity and definition in this process, imagine how endlessly complicated music would be if we considered every octave of a pitch to be its own unique entity (as it appears that the ancient Greeks did).[71]

With this notion of octave equivalence of course comes some rather significant abstractions about the concept of interval as well. If we say, for example, that all Cs are congruent and that all Ds are congruent, then this also means that all Cs lie a congruent "interval" from all Ds. Thus, we are saying that an ascending major second is somehow congruent to a descending minor seventh and any compound version of these intervals. Once again, it may be difficult to actually perceive this congruence in many cases, but this does not prevent us from assuming it in our analyses. To claim, for example, that two tone rows can only be considered the same if their interval content is *exactly* the same and they are played in the same octave would completely undermine our understanding of twelve-tone music. In fact, it would be almost impossible to observe any repetition at all in non-tonal music without the notion of octave equivalence and pitch-class interval.

Perhaps even more relevant to our discussion of the SUM classes are the assumptions inherent within the concept of a set class. To say that two sets belong to the same set class is to say that their total interval content is "the same." This seems a very natural means of classification when two sonorities or motives are played in the same register and ordering like we see in the first part of Example 4.5, but I would venture to guess that most people would have a much harder time hearing the two sonorities in the second part Example 4.5 as "the same," though their interval content is also identical. Moreover, a set and its inversion also have the same interval content, which means that all four sonorities in Example 4.5 are members of the

[71] See the fifth chapter of Mathiesen, *Apollo's Lyre*, 498–607. Particularly Figure 104 on page 599.

same set class. Once again, to limit our concept of a set class to only what was very easily perceptible would severely limit our analytical horizons and obscure many important relationships, but we should not forget the many conceptual leaps we make in the process. Since pitch-class sets are already quite abstract, any system that models the voice-leading between them will necessarily have to be quite abstract as well.



Example 4.5. Four different pitch-class sets from 3-3.

I discuss these assumptions and abstractions inherent within the pitch-class and pitch-class-set universes because I wish to show that many of our analytical systems for non-tonal music (and even tonal music) are based upon generalizations and abstracts that are not always easy or even possible to hear, and yet this has not prevented us from using them. Furthermore, many of these assumptions are not altogether different from those that the SUM-class system is founded upon. Consider, for example, that an octave is a “smaller” pitch-class interval than a semitone within pitch-class space. From this it is easy to see why voice leading in which the total voice-leading interval adds up to an octave receives a smaller PVLS value than voice leading in which a single voice moves by semitone. Thus, the SUM-class system merely extends an abstraction already present within the pitch-class universe.

The only assumption that is really unique to the SUM-class system is that voice leading in multi-part music is additive, meaning that the total interval between two sonorities is the sum of the individual intervals traversed in each voice. I am not necessarily sure that this is perceptible with the precision that we have often discussed it here (would it really be possible to hear that C+ to E- is smaller than C+ to A- by one semitone?), but I do think that the voice leading between sonorities with common tones sounds smoother than between sonorities without common tones. In any case, whether or not these abstractions are perceptible or more or less extreme than the abstractions required of the pitch-class universe is not really the issue. Music theorists use abstract analytical systems because the value they provide outweighs any concern for loss of precision. The question, then, is only whether the power of the SUM-class system is worth the additional levels of abstraction it requires.

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Appendix A: Compound SUM-Class Spaces

Chapters 2 and 3 examined SUM-class systems within the context of a single set class. In pitch-class set theory, a set class is defined as the set of all unique pitch-class sets that are related to one another by some transposition or inversion. That is, two sets *a* and *b* belong to the same set class if there exists some T_n or I_n transformation such that $T_n(a) = b$ or $I_n(a) = b$ and vice versa. By definition, then, pitch-class sets in different set classes cannot be transformed into each other by any T_n, I_n, or contextual transformation, which means that the kinds of SUM-class systems we have examined so far can only work within the context of a single set class. Though we have certainly seen examples of music in which a single set class is featured prominently, such examples are the exception rather than the rule. In order to study voice leading in other contexts, then, we need a way to expand the SUM-class system so that it can accommodate sets from different set classes, and to do this, we will need to define a set of transformations designed to move between sets in different set classes.

One such set of transformations are the so-called "multiplicative" transformations, which work by multiplying each pitch class within a pitch-class set by a given integer and returning a new set of these value modulo 12.[72] Formally:

Definition A.1. $M_n(\{x_1, x_2, \ldots x_i\}) = \{(x_1 n), (x_2 n), \ldots (x_i n)\}$.

[72] See Charles Wuorinen, *Simple Composition* (New York: Longman, 1979), 98–101.

Because of the way that multiplication interacts with modular arithmetic, many of these transformations behave in rather unusual (and not particularly useful) ways, as can be seen in Table A.1. What each of these transformations does, in essence, is to map the pitch classes of the chromatic scale to various symmetrical pitch structures.

{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11}	→ M_0	{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}
{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11}	⇔ M_1	{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10,11}
{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11}	→ M_2	{0, 2, 4, 6, 8, 10, 0, 2, 4, 6, 8, 10}
{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11}	→ M_3	{0, 3, 6, 9, 0, 3, 6, 9, 0, 3, 6, 9}
{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11}	→ M_4	{0, 4, 8, 0, 4, 8, 0, 4, 8, 0, 4, 8}
{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11}	⇔ M_5	{0, 5, 10, 3, 8, 1, 6, 11, 4, 9, 2, 7}
{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11}	→ M_6	{0, 6, 0, 6, 0, 6, 0, 6, 0, 6, 0, 6}
{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11}	⇔ M_7	{0, 7, 2, 9, 4, 11, 6, 1, 8, 3, 10, 5}
{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11}	→ M_8	{0, 8, 4, 0, 8, 4, 0, 8, 4, 0, 8, 4}
{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11}	→ M_9	{0, 9, 6, 3, 0, 9, 6, 3, 0, 9, 6, 3}
{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11}	→ M_{10}	{0, 10, 8, 6, 4, 2, 0, 10, 8, 6, 4, 2}
{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11}	⇔ M_{11}	{0, 11, 10, 9, 8, 7, 6, 5, 4, 3, 2, 1}

Table A.1. Mappings from the chromatic scale to various other pitch structures via the M_n transformations.

As can be seen, M_1, M_5, M_7, and M_{11} are the only M_n transformations to return all twelve pitch classes when applied to the chromatic scale whereas all other transformations just return the same set over and over. Furthermore, M_1, M_5, M_7, and M_{11} are the only M_n transformation that are their own inverses. Table A.1 also reveals, however, that M_1 and M_{11} produce the same mappings as T_0 and I_0 respectively (at least when applied to the whole chromatic scale), and so only M_5 and M_7 will be particularly useful for our present context.

In most cases, the M_5- or M_7-transform of a pitch-class set will actually still belong to the same set class as the original set. This occurs whenever a set has the same spacing within the chromatic as on a cycle of fourths (M_5) or fifths (M_7). Such set classes can be said to be "multiplicatively symmetrical." In some cases, however, M_5 and M_7 will actually map between sets belonging to different set classes. For example, $M_5(\{0, 1, 4\}$ (a set in 3-3) = $\{0, 5, 8\}$ (a set in 3-11) and $M_7(\{0, 1, 4\}) = \{0, 7, 4\}$ (a different set in 3-11).[73] When this happens and there is also a single group of transformations that acts simply transitively upon *both* set classes, we can then compose the multiplicative transformations with this other group of transformations to make a larger transformational group that will act simply transitively on the collective space of the two set classes. Furthermore, if this new compound group also possesses a normal subgroup that produces consistent voice-leading intervals on both set classes, then it is a simple matter to create a generalized voice-leading system for these two set classes that would be analogous to the systems for single set classes we saw in Chapters 1 and 2.

Exactly such a construction can be created for the M-related set classes 3-1 and 3-9 by expanding the equivalence relation that created these spaces so as to include the sets from both

[73] Just because both M_5 and M_7 map between the same set classes, however, does not mean that they are "the same." M_5 maps the chromatic scale to the circle of fourths whereas M_7 maps the chromatic scale to the circle of fifths. In a way, then, these two transformations are inversions of one another, and, indeed, in a non-symmetrical set class, the M_5- and M_7-transforms of the same set will be inversions of each other: $M_5(\{0, 1, 4\}) = \{0, 5, 8\}$, $M_7(\{0, 1, 4\}) = \{0, 7, 4\}$, and $I_0(\{0, 5, 8\}) = \{0, 7, 4\}$. These transformations will also tend to produce different voice-leading intervals between the sets they map, and so we will often have cause to choose one or the other in a different context.

classes. Since these sets generate the same SUM-class spaces, each SUM class in this new "compound" space (see Table A.2) contains three sets from each set class. These sets in each SUM class that belong to the same set class are always related to one another by T_0, T_4, or T_8 (as we saw in Chapter 3), and since M_5 maps between sets within the same SUM class in this system, we can see that sets in the same SUM class but from different set classes will always be related by T_0M_5, T_4M_5, or T_8M_5. For example, $T_0M_5(\{11, 0, 1\}) = \{5, 7, 0\}$, $T_4M_5(\{11, 0, 1\}) = \{9, 11, 4\}$, $T_8M_5(\{11, 0, 1\}) = \{1, 3, 8\}$. Through the combined forces of these compound transformations and T_0, T_4, and T_8, then, we can map between any two sets within a single SUM class. This should sound familiar, for this is exactly how the normal subgroups of the T_n/I_n and neo-Riemannian groups acted upon the sets of a single SUM class. The question, then, is whether the set of T_nM_5 and T_n transformations might not also form their own group. To examine this, let us consider the binary composition of these transformations (with $M_n = M_5$ or M_7) as summarized in Table A.3.

SUM Class	**Pitch-Class Set Members**
0	{{11, 0, 1}, {3, 4, 5}, {7, 8, 9}, {1, 3, 8}, {5, 7, 0}, {9, 11, 4}}
3	{{0, 1, 2}, {4, 5, 6}, {8, 9, 10}, {2, 4, 9}, {6, 8, 1}, {10, 0, 5}}
6	{{1, 2, 3}, {5, 6, 7}, {9, 10, 11}, {3, 5, 10}, {7, 9, 2}, {11, 1, 6}}
9	{{10, 11, 0}, {2, 3, 4}, {6, 7, 8}, {0, 2, 7}, {4, 6, 11}, {8, 10, 3}}

Table A.2. The SUM classes of the compound space for 3-1/3-9.

	T_m	$T_m M_x$
T_n	T_{m+n}	$T_{(m+n)}M_x$
$T_n M_x$	$T_{(n+xm)}M_x$	T_{n+xm}

Table A.3. The binary composition of the T_n and T_nM_x transformations.

As can be seen, since the binary composition deals in mod-12 arithmetic, any *n* + *m* or *n* + *xm* seen in Table A.3 will always be another mod-12 integer and thus a transformation in the group. Similarly, $(T_a \circ T_bM_x) \circ T_cM_x = T_{(a+b)}M_x \circ T_cM_x = T_{(a+b)+xc}$; $T_a \circ (T_bM_x \circ T_cM_x) = T_a \circ T_{(b+xc)} = T_{(b+xc)+a}$ and clearly $T_{(a+b)+xc} = T_{(b+xc)+a}$ because mod-12 multiplication and addition are associative. We already know that the inverse of any T_n is T_{12-n} and it should be easy to see that the inverse of a T_nM_5 transformation is $T_{12-xn}M_5$. Finally, the group identity is T_0, and so the twelve T_n and twelve T_nM_x transformations together form a group. The set $\{T_0, T_4, T_8, T_0M_5, T_4M_5, T_8M_5\}$ that we discussed earlier is a normal subgroup of this T_n/T_nM_5 group, and it should not surprise us that the cosets of this subgroup are isomorphic to the *Zn* transformations that act on the SUM-class spaces of both 3-1 and 3-9 (see Table A.4). Both of these transformations act simply transitively on their respective spaces and as such can define GISs. Together, these two GISs allow us to create fully-functional generalized voice-leading systems for two entirely different symmetrical set classes!

T_n and T_nM_5 Transformations	Isomorphism	*Zn* Transformations
{T_0, T_4, T_8, T_0M_5, T_4M_5, T_8M_5}	⇔	Z0
{T_1, T_5, T_9, T_1M_5, T_5M_5, T_9M_5}	⇔	Z3
{T_2, T_6, T_{10}, T_2M_5, T_6M_5, $T_{10}M_5$}	⇔	Z6
{T_3, T_7, T_{11}, T_3M_5, T_7M_5, $T_{11}M_5$}	⇔	Z9

Table A.4. The homomorphism from the T_n/T_nM_5 group onto the order-four *Zn* group mediated by the isomorphism between the quotient group of the T_n/T_nM_5 group modulo {T_0, T_4, T_8, T_0M_5, T_4M_5, T_8M_5} and the order-four *Zn* subgroup.

But what about M-related set classes that are *not* symmetrical? Is this same sort of construction also possible for them as well? Not only will there be twice as many pitch-class sets within these spaces, but we will also have to deal with the prime and inverted forms of each set class and transformations to move between them. Let us investigate this in the context of the compound SUM-class system for 3-2/3-7 seen in Table A.5.

SUM Class	Pitch-Class Set Members
1	{{3, 4, 6}, {7, 8, 10}, {11, 0, 2}, {2, 4, 7}, {6, 8, 11}, {10, 0, 3}}
2	{{11, 1, 2}, {3, 5, 6}, {7, 9, 10}, {10, 1, 3}, {2, 5, 7}, {6, 9, 11}}
4	{{0, 1, 3}, {4, 5, 7}, {8, 9, 11}, {3, 5, 8}, {7, 9, 0}, {11, 1, 4}}
5	{{0, 2, 3}, {4, 6, 7}, {8, 10, 11}, {7, 10, 0}, {11, 2, 4}, {3, 6, 8}}
7	{{1, 2, 4}, {5, 6, 8}, {9, 10, 0}, {0, 2, 5}, {4, 6, 9}, {8, 10, 1}}
8	{{9, 11, 0}, {1, 3, 4}, {5, 7, 8}, {8, 11, 1}, {0, 3, 5}, {4, 7, 9}}
10	{{2, 3, 5}, {6, 7, 9}, {10, 11, 1}, {1, 3, 6}, {5, 7, 10}, {9, 11, 2}}
11	{{10, 0, 1}, {2, 4, 5}, {6, 8, 9}, {9, 0, 2}, {1, 4, 6}, {5, 8, 10}}

Table A.5. The SUM classes of the compound space for 3-2/3-7.

As before, the three sets in each SUM class that are members of the same set class will be related to one another by either T_0, T_4, or T_8. Here, however, M_5 no longer maps between sets in the same SUM class but between opposite-quality sets that are members of different set classes (and for these two sets, opposite quality sets are always in different SUM classes). M_5 of the prime-form representative of 3-2 ({0, 1, 3}), for example, takes us to the inverted-form representative of 3-7 ({0, 3, 5}) and vice versa. The M_7 transformation, on the other hand, maps between same-quality sets from different set classes: M_7({0, 1, 3}) = {7, 9, 0}—also a prime form of 3-7. Thus, the M_7 transformation is something like a "transposition" between the two set classes and M_5 something like an inversion. But these two transformations and the T_n transformations will not be enough to move between *every* set in the space, because they do not provide a way of moving between inversionally-related sets within the same set class. For these mappings, we will need either the I_n or generalized neo-Riemannian transformations. Once we have invoked these inversions, there will be no need to make use of both M_5 and M_7 since the product of an inversion with one of these transformations will produce the same mappings as the other transformation. That is, there is some I_nM_5 transformation that has the same effect as M_7 for every set and vice versa, and so using M_5, M_7, and an inversion in the same group would cause the group to not be simply transitive since there would always be at least two ways to get from set *a* to set *b*. In total, then, to move between all of the sets in the compound 3-2/3-7 space will require a set of transpositions, a set of inversions, a set of transpositions composed with a multiplicative transformation, and a set of inversions composed with a multiplicative transformation.

One such set of transformations is the order-forty-eight set of twelve T_n, twelve I_n, twelve T_nM_x, and twelve I_nM_x transformations whose binary composition is sketched in Table A.6. The table reveals that the group is indeed closed under the binary composition and of course T_0 will still be the group identity. We already know that every T_n, I_n, and T_nM_x will have its inverse in the group, and it can similarly be seen that the inverse of an I_nM_x transformation will be the transformation I_mM_x transformation such that $mx = n$. As an example of the group's associativity: $(I_1M_5 \circ I_2M_5) \circ I_3M_5 = T_{1-10} \circ I_3M_5 = T_3 \circ I_3M_5 = I_{(3+3)}M_5 = I_6M_5$ and likewise, $I_1M_5 \circ (I_2M_5 \circ$

$I_3M_5) = I_1M_5 \circ T_{(2-(3)(5))} = I_1M_5 \circ T_{11} = I_{(1-(11)(5))}M_5 = I_{(1-7)}M_5 = I_6M_5$. The set thus meets all criteria of a group.

	T_m	T_mM_x	I_m	I_mM_x
T_n	T_{m+n}	$T_{(m+n)}M_x$	I_{m+n}	$I_{(m+n)}M_x$
T_nM_x	$T_{(n+xm)}M_x$	T_{n+xm}	$I_{(n+xm)}M_5$	I_{n+xm}
I_n	I_{n-m}	$I_{(n-m)}M_x$	T_{n-m}	$T_{(n-m)}M_x$
I_nM_x	$I_{(n-xm)}M_x$	I_{n-xm}	$T_{(n-xm)}M_5$	T_{n-xm}

Table A.6. The binary composition of the T_n, I_n, T_nM_x, and I_nM_x transformations where *m* and *n* = the mod-12 integers 0–11 and $x = 5$ or 7.

Since M_x on the group table above could be either M_5 or M_7 without altering the structure of the group, is there any reason to choose one transformation over the other? Recall that our ultimate goal is to be able to create a generalized voice-leading system for this compound space, and toward that end we would of course want these transformations on their own to produce consistent voice leading. By examining the effect of M_5 and M_7 at the level of the SUM classes, we can see that M_5 always maps between sets that lie the same voice-leading interval from one another (($\boxed{1}$, $\boxed{5}$), ($\boxed{2}$, $\boxed{10}$), ($\boxed{4}$, $\boxed{8}$), ($\boxed{7}$, $\boxed{11}$)) whereas M_7 sometimes maps between sets in the same SUM class and sometimes in the most distant SUM classes (($\boxed{1}$, $\boxed{7}$), ($\boxed{2}$) ($\boxed{4}$) ($\boxed{5}$, $\boxed{11}$), ($\boxed{8}$) ($\boxed{10}$)). M_5 will thus be the most useful transformation for our current context.

Collectively, the T_n/I_n/T_nM_5/I_nM_5 transformation group makes it possible to move from any pitch-class set in 3-2 to any pitch-class set in either 3-2 or 3-7 and vice versa. Sets of the same quality in the same set class will be related by some T_n, sets of opposite quality in the same set class will be related by some I_n, sets of opposite quality in different set classes will be related by T_nM_5, and sets of the same quality in different set classes will be related by I_nM_5. In other words, this group acts simply transitively on the combined space of 3-2/3-7 and can thus form a GIS. The set $\{T_0, T_4, T_8, T_0M_5, T_4M_5, T_8M_5\}$ is still a normal subgroup of this larger order-forty-eight group, but the cosets of this subgroup do not produce consistent motion at the level of the SUM class. The I_n transformations from the $\{I_0, I_4, I_8, I_0M_5, I_4M_5, I_8M_5\}$ coset, for example, will map any set in $\boxed{1}$ to a set in $\boxed{11}$ whereas the I_nM_5 transformations map sets in $\boxed{1}$ to sets in $\boxed{7}$. The problem here is not with the transformations but the cosets—I_n and I_nM_5 simply need to be segregated from one another. Using the $\{T_0, T_4, T_8\}$ normal subgroup to generate cosets instead produces the sixteen unique cosets seen in the left column of Table A.7 in which, among other things, the I_n and I_nM_5 transformations are separated from one another. The right column of the table displays the mappings these cosets produce at the SUM-class level when they are applied to the pitch-class sets of 3-2/3-7.

{T_0, T_4, T_8} Cosets	**Secondary Mappings at the SUM-Class Level**
{T_0, T_4, T_8}	$(\boxed{1})\ (\boxed{2})\ (\boxed{4})\ (\boxed{5})\ (\boxed{7})\ (\boxed{8})\ (\boxed{10})\ (\boxed{11})$
{T_1, T_5, T_9}	$(\boxed{1}, \boxed{4}, \boxed{7}, \boxed{10})\ (\boxed{2}, \boxed{5}, \boxed{8}, \boxed{11})$
{T_2, T_6, T_{10}}	$(\boxed{1}, \boxed{7})\ (\boxed{2}, \boxed{8})\ (\boxed{4}, \boxed{10})\ (\boxed{5}, \boxed{11})$
{T_3, T_7, T_{11}}	$(\boxed{1}, \boxed{10}, \boxed{7}, \boxed{4})\ (\boxed{2}, \boxed{11}, \boxed{8}, \boxed{5})$
{I_0, I_4, I_8}	$(\boxed{1}, \boxed{11})\ (\boxed{2}, \boxed{10})\ (\boxed{4}, \boxed{8})\ (\boxed{5}, \boxed{7})$
{I_1, I_5, I_9}	$(\boxed{1}, \boxed{2})\ (\boxed{4}, \boxed{11})\ (\boxed{5}, \boxed{10})\ (\boxed{7}, \boxed{8})$
{I_2, I_6, I_{10}}	$(\boxed{1}, \boxed{5})\ (\boxed{2}, \boxed{4})\ (\boxed{7}, \boxed{11})\ (\boxed{8}, \boxed{10})$
{I_3, I_7, I_{11}}	$(\boxed{1}, \boxed{8})\ (\boxed{2}, \boxed{7})\ (\boxed{4}, \boxed{5})\ (\boxed{10}, \boxed{11})$
{T_0M_5, T_4M_5, T_8M_5}	$(\boxed{1}, \boxed{5})\ (\boxed{2}, \boxed{10})\ (\boxed{4}, \boxed{8})\ (\boxed{7}, \boxed{11})$
{T_1M_5, T_5M_5, T_9M_5}	$(\boxed{1}, \boxed{8}, \boxed{7}, \boxed{2})\ (\boxed{4}, \boxed{11}, \boxed{10}, \boxed{5})$
{T_2M_5, T_6M_5, $T_{10}M_5$}	$(\boxed{1}, \boxed{11})\ (\boxed{2}, \boxed{4})\ (\boxed{5}, \boxed{7})\ (\boxed{8}, \boxed{10})$
{T_3M_5, T_7M_5, $T_{11}M_5$}	$(\boxed{1}, \boxed{2}, \boxed{7}, \boxed{8})\ (\boxed{4}, \boxed{5}, \boxed{10}, \boxed{11})$
{I_0M_5, I_4M_5, I_8M_5}	$(\boxed{1}, \boxed{7})\ (\boxed{2})\ (\boxed{4})\ (\boxed{5}, \boxed{11})\ (\boxed{8})\ (\boxed{10})$
{I_1M_5, I_5M_5, I_9M_5}	$(\boxed{1}, \boxed{10})\ (\boxed{2}, \boxed{5})\ (\boxed{4}, \boxed{7})\ (\boxed{8}, \boxed{11})$
{I_2M_5, I_6M_5, $I_{10}M_5$}	$(\boxed{1})\ (\boxed{2}, \boxed{8})\ (\boxed{4}, \boxed{10})\ (\boxed{5})\ (\boxed{7})\ (\boxed{11})$
{I_3M_5, I_7M_5, $I_{11}M_5$}	$(\boxed{1}, \boxed{4})\ (\boxed{2}, \boxed{11})\ (\boxed{5}, \boxed{8})\ (\boxed{7}, \boxed{10})$

Table A.7. The cosets of {T_0, T_4, T_8} in the $T_n/I_n/T_nM_5/I_nM_5$ group and the secondary mappings they produce at the SUM-class level.

We have already observed (see Chapter 3) the SUM-class-level mappings (i.e., voice-leading intervals) produced secondarily by the T_n and I_n cosets when they are applied to the sets of either 3-2 or 3-7, and since these transformations move only *within* set classes, these mappings will not change here. The T_nM_5 transformations, on the other hand, not only behave very differently here than they did for 3-1/3-9, but also produce highly inconsistent and idiosyncratic voice-leading intervals—as do the new I_nM_5 transformations. In all of the GISs we have created thus far, we have always used the transformations from the T_n/I_n group and the inversions from the contextual group to create a set of transformations that produced consistent voice-leading intervals. Since the T_n transformations do not produce consistent voice-leading intervals here, it is not clear that such a construction would be possible. Thus, while we can certainly define a simply-transitive group of transformations for these non-symmetrical compound spaces, it may not be possible to use them to generalize voice-leading intervals within the space as it was for the symmetrical spaces. Further research is needed in this area.

Note that all M-related trichords we have examined so far have generated the same SUM-class spaces. We can show that this will always be the case because $M_n(\{a, b, c\}) = \{na, nb, nc\}$. In the context of a SUM-class system, sets are grouped together according to the sum of their constituent pitch classes. Therefore, $\{a, b, c\}$ will belong to SUM class $(a + b + c)$ and the M_n-transform of $\{a, b, c\}$ will belong to SUM class $(na + nb + nc)$, which may be simplified algebraically to $(n)(a + b + c)$. What this shows us is that the result of summing an M-transformed set is the same as M-transforming the sum of a set. In other words, the M_n-transform of a set in SUM class x will be a set in SUM class nx (mod 12, of course). Because of this, we can also see that a set class that generates the SUM-class space $\{\boxed{a}, \boxed{b}, \boxed{c}\}$ will be M_n related to a set class that generates the SUM-class space $\{\boxed{na}, \boxed{nb}, \boxed{nc}\}$. Any set class with a SUM-class space whose SUM classes map to themselves under M_n (pretending for a moment that the SUM classes were really pitch classes upon which M_n is actually defined to act) will thus be M_n related to a set class with the same SUM-class space. It just so happens that *all* of the possible SUM-class spaces for any cardinality map to themselves under both M_5 and M_7. Consider, for example, the two trichord spaces $\{\boxed{0}, \boxed{3}, \boxed{6}, \boxed{9}\}$ and $\{\boxed{1}, \boxed{2}, \boxed{4}, \boxed{5}, \boxed{7}, \boxed{8}, \boxed{10}, \boxed{11}\}$: $M_5(\{\boxed{0}, \boxed{3}, \boxed{6}, \boxed{9}\}) = \{\boxed{0}, \boxed{3}, \boxed{6}, \boxed{9}\}$, $M_7(\{\boxed{0}, \boxed{3}, \boxed{6}, \boxed{9}\}) = \{\boxed{0}, \boxed{9}, \boxed{6}\ \boxed{3}\}$, $M_5(\{\boxed{1}, \boxed{2}, \boxed{4}, \boxed{5}, \boxed{7}, \boxed{8}, \boxed{10}, \boxed{11}\}) = \{\boxed{5}, \boxed{10}, \boxed{8}, \boxed{1}, \boxed{11}, \boxed{4}, \boxed{2}, \boxed{7}\}$, and $M_7(\{\boxed{1}, \boxed{2}, \boxed{4}, \boxed{5}, \boxed{7}, \boxed{8}, \boxed{10}, \boxed{11}\}) = \{\boxed{7}, \boxed{2}, \boxed{4}, \boxed{11}, \boxed{1}, \boxed{8}, \boxed{10}, \boxed{5}\}$. While the ordering of these sets is scrambled, the SUM classes contained within them remain the same. Thus, we can know that any two set classes related to one another by M_5 or M_7 will always generate the same SUM classes. Similarly, we can also know that no SUM-class space outlining the full chromatic scale (the pentachords and heptachords) will produce consistent voice leading under M_5 or M_7 because we saw earlier that these transformations map the chromatic scale to the circles of fourths and fifths respectively, and each point on the chromatic scale does not lie the same distance from each point on the circle of fourths or fifths.

We noted in Chapter 3 that complementary set classes always generated the same SUM-class spaces, and we have also seen that M-related sets always generate the same SUM-class spaces as well. Does this imply that the complements of two M-related sets will also be M-related? To investigate this, let us once again recall the action of M_5 and M_7 on the chromatic scale, which can be seen in Tables A.8 and A.9.

	$\mathbf{M_5}$	
0	⇔	0
1	⇔	5
2	⇔	10
3	⇔	3
4	⇔	8
5	⇔	1
6	⇔	6
7	⇔	11
8	⇔	4
9	⇔	9
10	⇔	2
11	⇔	7

Table A.8. The mapping table for M_5 on the pitch classes. Lines connect complements.

	$\mathbf{M_7}$	
0	⇔	0
1	⇔	7
2	⇔	2
3	⇔	9
4	⇔	4
5	⇔	11
6	⇔	6
7	⇔	1
8	⇔	8
9	⇔	3
10	⇔	10
11	⇔	5

Table A.9. The mapping table for M_7 on the pitch classes. Lines connect complements.

What the lines on these tables show us is that complementary pitch classes are always mapped to pitch classes that are *also* complementary under both M_5 and M_7. In other words, the complement relation is preserved under M_5 and M_7. This means that a set and its complement will both be M-related to set and its complement, or, conversely, that the complements of two M-related sets will also be M-related. It should be obvious, then, that the same would be true of entire set classes, meaning that if 3-3 and 3-11 are M-related, then 9-3 and 9-11 will also be M-related. Because of this, we can know that any observations we make for compound SUM-class spaces for sets of cardinalities 3, 4, and 5 will also be true for sets of cardinalities 9, 4, and 7 respectively—the only difference being (as we noted in Chapter 3) that the T_n transformations will create complementary voice-leading distances for complementary set classes. In short, complementary compound SUM-class spaces are no more different from each other than the spaces of the complementary set classes themselves.

Appendix B: A Super SUM-Class System for all Pitch-Class Sets

In Chapter 4 we saw that the overall relationship between SUM classes and voice-leading intervals was unaffected by the cardinality of the sets the SUM-classes were defined upon. That is, it was always true that the PVLS between two sets *a* and *b* is always equal to SUM(*b*) – SUM(*a*) regardless of whether sets *a* and *b* contain two pitch classes or eleven pitch classes. This suggests that the SUM-class system might also be able to generalize voice-leading intervals across cardinality, and if this were possible, then we could create one massive super SUM-class system for all pitch-class sets. Clearly, however, none of the pitch-class set transformations we have worked with in this thesis will be able to transform a set from one cardinality into a set of a different cardinality. Moreover, even if we were to try to create a set of cross-cardinality transformations, we would have to define as many variations of these transformations as there are pitch-class sets in order to create a simply transitive group.

A much more manageable task would be to create a system for all pitch-class sets in the vein of Chapter 4, in which we abandon the idea of pitch-class-set transformations altogether and focus only on the SUM classes and their relationship to the pitch-class sets they contain. We begin by creating the super SUM-class space itself:

Definition B.1. Let *S* be the set of all possible pitch-class sets of cardinalities zero to twelve and *R* a relation on *S* such that $(a, b) \in R$ for any $a, b \in S$ that satisfy the equation SUM(*a*) = SUM(*b*).

This creates a structure identical to the super SUM-class systems of Chapter 4, except now sets of different cardinalities inhabit the same SUM classes. The Z*n* group will of course still act simply transitively upon the SUM-classes of this space since they have not changed in any way. In order to relate these Z*n* transformations to voice-leading intervals between pitch-class sets as we did in Chapter 4, however, we will need to redefine PVLS so that it is not cardinality-dependent.

Recall that PVLS is defined so as to calculate the difference between all ordered pairs of pitch-classes within two sets and then return a SUM of these differences. Formally:

> **Definition B.2.** Let *X* and *Y* be pitch-class sets of cardinality *z* of the form $\{x_1, x_2, \ldots x_z\}$ and $\{y_1, y_2, \ldots y_z\}$ and let a pairwise voice-leading sum from *X* to *Y* (written as PVLS(*X*,*Y*)) = $\sum_{n=1}^{z}(y_n - x_n)$ modulo 12.

As this stands now, there is no meaningful way of calculating PVLS between two sets of different size because this would leave pitch classes in the larger set with no pitch class against which to calculate its difference. One way of working around this would be to equalize the cardinalities of the two sets in some way. This could be done by either deleting pitch classes from the larger set or by adding pitch classes to the smaller set. Certainly, deletion could produce interesting results, but the problem would be deciding which pitch classes to delete in any sort of systematic, nonarbitrary way. In any case, deleting pitch classes from a set would obviously change its SUM value and thus its SUM-class membership, which would obscure the relationship between SUM class and voice leading.

If we were to equalize the cardinalities of the two sets by adding zeros to the smaller set, however, this would *not* change the SUM value of the smaller set. To calculate PVLS between {7, 8, 10, 11, 0, 3} and {11, 1, 3} (both members of $\boxed{0}$), for example, we could temporarily add three zeros to {11, 1, 3} so that the two sets both have cardinality 6. PVLS({{7, 8, 10, 11, 0, 3}, {11, 1, 3, 0, 0, 0}) is then = ((11– 7) + (1 – 8) + (3 – 10) + (0 – 11) + (0 – 0) + (0 – 3)) = 0, and this still preserves the relationship between SUM class and PVLS values because both of these sets are members of the same SUM class ($\boxed{3}$). Thus, we see that this "zero-packing" method allows us to still calculate voice-leading intervals by using PVLS and also preserves the relationship between PVLS and SUM value. In other words, we could say that the voice-leading interval between any two sets in the super SUM-class system for all pitch-class sets will be equal to *n* of the Z*n* transformation that moves between the SUM classes these sets inhabit. That is, the interval from *a* to *b* = $Z_{\mathrm{PVLS}(\mathrm{SUM}(a),\, \mathrm{SUM}(b))} = Z_{(\mathrm{SUM}(b) - \mathrm{SUM}(a))}$—just as we saw in Chapter 4 for the single-cardinality systems.

Clearly there are significant philosophical issues with the zero-packing method, however, because adding zeros to a pitch-class set literally means that we are adding Cs. When we measure PVLS in this way, then, we are really measuring the voice-leading interval from a larger set to a smaller set with some added Cs. It is questionable, then, what this really tells us about the voice-leading interval between the larger set and the smaller set. Indeed, it is actually rather difficult to say what it means to voice lead between sets of different cardinalities anyway.[74] While this sketch is obviously quite problematic, I think there is still something suggestive about the notion that SUM classes capture voice-leading intervals regardless of the cardinalities of the sets contained within them. More work would clearly need to be done in this area.

[74] See Joti Rockwell, "Birdcage Flights: A Perspective on Inter-Cardinality Voice Leading," *Music Theory Online* 15, no. 5 (2009).